Physics 504, Lecture 7
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1 Sources of Waves in Wave Guides

We now turn our attention to the sources of waves in waveguides. With given sources (*i.e.* ignoring back reactions) the equations are still linear (inhomogeneous) in the fields and time-independent, so we can assume everything has a $e^{-i\omega t}$ time dependence. The normal modes of free waves in a waveguide form a complete set of states for the source-free solutions to Maxwell's equations in the guide, though we need to include all the modes, including those whose cutoff frequency is above ω. Far from the sources, however, only the real values of k, for modes with $\omega_\lambda < \omega$, will have appreciable amplitude.

We will expand our fields in the normal modes, which will be indexed by a composite index λ, which includes information about whether the mode is TE or TM (or of other nature, *e.g.* TEM), and also the indices which specify which mode (roughly the angular and radial 'quantum' numbers) For a given mode we have two values of k, either positive (right moving) and negative (left moving), or $\pm i|k|$ for cutoff modes. We will lump the $+i|k|$ modes in with the positive k ones, and write the $(+)$ part of the λ mode as

$$\begin{aligned}
\vec{E}^+_\lambda(x,y,z) &= \left[\vec{E}_\lambda(x,y) + \hat{z}E_{z\,\lambda}(x,y)\right] e^{ik_\lambda z} \\
\vec{H}^+_\lambda(x,y,z) &= \left[\vec{H}_\lambda(x,y) + \hat{z}H_{z\,\lambda}(x,y)\right] e^{ik_\lambda z}
\end{aligned}$$

where $\vec{E}$ and $\vec{H}$ are purely transverse, and were given in terms of E_z or H_z for TM or TE modes respectively earlier. There are also modes travelling in the negative z direction. The equations are symmetric under $z \leftrightarrow -z$, under which the transverse $\vec{E}_\lambda(x,y)$ is unchanged but $E_{z\,\lambda}(x,y)$ changes sign. The magnetic field is a *pseudo*-vector, so for it the transverse $\vec{H}_\lambda(x,y)$ changes sign but $H_{z\,\lambda}(x,y)$ is unchanged. Thus

$$\begin{aligned}
\vec{E}^-_\lambda(x,y,z) &= \left[\vec{E}_\lambda(x,y) - \hat{z}E_{z\,\lambda}(x,y)\right] e^{-ik_\lambda z} \\
\vec{H}^-_\lambda(x,y,z) &= \left[-\vec{H}_\lambda(x,y) + \hat{z}H_{z\,\lambda}(x,y)\right] e^{-ik_\lambda z}
\end{aligned}$$

The transverse fields $\vec{E}_\lambda$ form a basis which we can choose to be real and normalized,

$$\int_A \vec{E}_\lambda \cdot \vec{E}_\mu = \delta_{\lambda\mu}.$$

That this can be done, and what this normalization requires for the basis of normal modes for E_z and H_z is elaborated in the slides. With this normalization, and from $\vec{H}_\lambda = Z_\lambda^{-1}\hat{z} \times \vec{E}_\lambda$ we also have

$$\int_A \vec{H}_\lambda \cdot \vec{H}_\mu = \frac{1}{Z_\lambda^2}\delta_{\lambda\mu},$$

and in the time-averaged power expression $\langle P\rangle = \frac{1}{2}\int_A \left(\vec{E} \times \vec{H}\right) \cdot \hat{z}$ we have

$$\int_A \left(\vec{E}_\lambda \times \vec{H}_\mu\right) \cdot \hat{z} = \frac{1}{Z_\lambda}\delta_{\lambda\mu}$$

The above normalization for the $\vec{E}_\lambda$ comes from requiring that the z components satisfy orthogonality conditions and normalization conditions adjusted appropriately. For TM waves, $\vec{E}_\lambda = ik_\lambda\gamma_\lambda^{-2}\vec{\nabla}E_{z\,\lambda}$, so

$$\begin{aligned}
\delta_{\lambda\mu} &= \int_A \vec{E}_\lambda \cdot \vec{E}_\mu = -\frac{k_\lambda k_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A \vec{\nabla}E_{z\,\lambda} \cdot \vec{\nabla}E_{z\,\mu} = \frac{k_\lambda k_\mu}{\gamma_\lambda^2\gamma_\mu^2}\int_A E_{z\,\lambda}\nabla^2 E_{z\,\mu} \\
&= -\frac{k_\lambda k_\mu}{\gamma_\lambda^2}\int_A E_{z\,\lambda}E_{z\,\mu},
\end{aligned}$$

where in the integration by parts (third =) the surface term vanishes as $E|_S = 0$, and for the fourth $\nabla^2 E_{z\,\mu} = -\gamma_\mu^2 E_{z\,\mu}$, so

$$\int_A E_{z\,\lambda}E_{z\,\mu} = -\frac{\gamma_\lambda^2}{k_\lambda^2}\delta_{\lambda\mu},$$

while for TE waves the same argument for H gives

$$\int_A H_{z\,\lambda}H_{z\,\mu} = -\frac{\gamma_\lambda^2}{Z_\lambda^2 k_\lambda^2}\delta_{\lambda\mu}.$$

For a rectangular waveguide, $0 \le x \le a \times 0 \le y \le b$, the modes are labelled by integers m and n, with

TM waves: $\psi|_S = 0$

$$\begin{aligned} E_{z\,mn} &= \psi = \frac{-2i\gamma_{mn}}{k_\lambda\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\ E_{x\,mn} &= \frac{2\pi m}{\gamma_{mn} a\sqrt{ab}} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\ E_{y\,mn} &= \frac{2\pi n}{\gamma_{mn} b\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \end{aligned}$$

TE waves: $\left.\frac{\partial\psi}{\partial n}\right|_S = 0$

$$\begin{aligned} H_{z\,mn} &= \psi = \frac{-2i\gamma_{mn}}{k_\lambda Z_\lambda\sqrt{ab}} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \\ E_{x\,mn} &= \frac{-2\pi n}{\gamma_{mn} b\sqrt{ab}} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right), \\ E_{y\,mn} &= \frac{2\pi m}{\gamma_{mn} a\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right), \end{aligned}$$

where

$$\gamma_{mn}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right).$$

The functional form of ψ is immediately apparent from the boundary conditions, and for TM modes $\vec{E} = ik\vec{\nabla}\psi/\gamma^2$, and for TE modes $\vec{E} = -iZk\hat{z} \times \vec{\nabla}\psi/\gamma^2$. The overall constants are determined from the normalization $\int_A E_x^2 + E_y^2 = 1$, except that for TE modes, we need an extra factor of $1/\sqrt{2}$ for each n or m which is zero, as $\int \cos^2(m\pi x/a) = a(1 + \delta_{m0})/2$.

1.1 Expansion of free waves

In our waveguide, an arbitrary field configuration in the absence of sources can be described by expanding in normal modes with positive and negative (or $+i$ and $-i$) wave numbers, as described above. Thus a total field

$$\vec{E} = \vec{E}^+ + \vec{E}^-, \qquad \vec{H} = \vec{H}^+ + \vec{H}^-,$$

with

$$E^{\pm} = \sum_{\lambda} A^{\pm}_{\lambda} \vec{E}^{\pm}_{\lambda}, \qquad H^{\pm} = \sum_{\lambda} A^{\pm}_{\lambda} \vec{H}^{\pm}_{\lambda},$$

can describe an arbitrary field configuration in a region that has no sources, with the A's constant (independent of z) in such a section of the wave guide. The coefficients are determined by the transverse components $\vec{E}$ and $\vec{H}$ along any cross section, for $\vec{E}$ has expansion coefficients $A^{+}_{\lambda} e^{ik_\lambda z} + A^{-}_{\lambda} e^{-ik_\lambda z}$ while $\vec{H}$ has expansion coefficients $A^{+}_{\lambda} e^{ik_\lambda z} - A^{-}_{\lambda} e^{-ik_\lambda z}$. This gives the expansion coefficients (taking $z = 0$) as

$$A^{\pm}_{\lambda} = \frac{1}{2} \int_A \vec{E} \cdot \vec{E}_\lambda \pm Z^2_\lambda \vec{H} \cdot \vec{H}_\lambda.$$

1.2 Localized Sources

We have described the waves that can propagate in the waveguide, but what actually produces such waves? We will consider a localized source with current density $\vec{J}(\vec{x})e^{-i\omega t}$ confined to some region $z \in [z_-, z_+]$, at the ends of which we imagine cross sections denoted by S_-, S_+. We will assume there are no sources or obstacles to the right of S_+ or to the left of S_-, so at S_+ there is no amplitude for any mode with negative k or with $-i|k|$, which would represent left-moving waves or exponential blow up (as $z \to +\infty$). The reverse is true at S_-, so

$$\begin{aligned}
\vec{E} &= \sum_{\lambda'} A^{+}_{\lambda'} \vec{E}^{+}_{\lambda'}, &\qquad \vec{H} &= \sum_{\lambda'} A^{+}_{\lambda'} \vec{H}^{+}_{\lambda'} &\qquad &\text{at } S_+ \\
\vec{E} &= \sum_{\lambda'} A^{-}_{\lambda'} \vec{E}^{-}_{\lambda'}, &\qquad \vec{H} &= \sum_{\lambda'} A^{-}_{\lambda'} \vec{H}^{-}_{\lambda'} &\qquad &\text{at } S_-
\end{aligned}$$

In between, we have the full Maxwell equations (with sources),

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = i\omega\mu_0 \vec{H}, \quad \vec{\nabla} \times \vec{H} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J} - i\omega\epsilon_0 \vec{E},$$

while the normal modes obey Maxwell equations without sources:

$$\vec{\nabla} \times \vec{H}^{\pm}_{\lambda} = -i\omega\epsilon_0 \vec{E}^{\pm}_{\lambda}, \quad \vec{\nabla} \times \vec{E}^{\pm}_{\lambda} = i\omega\mu_0 \vec{H}^{\pm}_{\lambda}.$$

If we apply the identity

$$\vec{\nabla} \cdot \left(\vec{A} \times \vec{B}\right) = \left(\vec{\nabla} \times \vec{A}\right) \cdot \vec{B} - \vec{A} \cdot \left(\vec{\nabla} \times \vec{B}\right),$$

we find

$$\begin{aligned}
&\vec{\nabla}\cdot\left(\vec{E}\times\vec{H}^{\pm}_{\lambda}-\vec{E}^{\pm}_{\lambda}\times\vec{H}\right)\\
&\quad=\left(\vec{\nabla}\times\vec{E}\right)\cdot\vec{H}^{\pm}_{\lambda}-\vec{E}\cdot\left(\vec{\nabla}\times\vec{H}^{\pm}_{\lambda}\right)-\left(\vec{\nabla}\times\vec{E}^{\pm}_{\lambda}\right)\cdot\vec{H}+\vec{E}^{\pm}_{\lambda}\cdot\left(\vec{\nabla}\times\vec{H}\right)\\
&\quad=i\omega\mu_0\vec{H}\cdot\vec{H}^{\pm}_{\lambda}+i\omega\epsilon_0\vec{E}\cdot\vec{E}^{\pm}_{\lambda}-i\omega\mu_0\vec{H}^{\pm}_{\lambda}\cdot\vec{H}+\vec{E}^{\pm}_{\lambda}\cdot\left(\vec{J}-i\omega\epsilon_0\vec{E}\right)\\
&\quad=\vec{J}\cdot\vec{E}^{\pm}_{\lambda}
\end{aligned}$$

If we integrate this over the volume between S_- and S_+, using Gauss' theorem and the boundary condition that $\vec{E}_{\|}=0$ at the conductor's surface,

$$\int_S\left(\vec{E}\times\vec{H}^{\pm}_{\lambda}-\vec{E}^{\pm}_{\lambda}\times\vec{H}\right)\cdot\hat{n}=\int_V\vec{J}\cdot\vec{E}^{\pm}_{\lambda},$$

where S consists of S_+ with $\hat{n}=\hat{z}$, and S_- with $\hat{n}=-\hat{z}$.

Let's take the upper sign. The contribution from S_+ is can be reduced to an integral over A at $z=0$:

$$\begin{aligned}
&\sum_{\lambda'}A^+_{\lambda'}\int_{S_+}\left(\vec{E}^+_{\lambda'}\times\vec{H}^+_{\lambda}-\vec{E}^+_{\lambda}\times\vec{H}^+_{\lambda'}\right)_z\\
&\quad=\sum_{\lambda'}A^+_{\lambda'}\int_{S_+}\left(\vec{E}_{\lambda'}\times\vec{H}_{\lambda}-\vec{E}_{\lambda}\times\vec{H}_{\lambda'}\right)_z e^{i(k_\lambda+k_{\lambda'})z}\\
&\quad=\sum_{\lambda'}A^+_{\lambda'}\int_{A}\left(\vec{E}_{\lambda'}\times\left(Z^{-1}_{\lambda}\hat{z}\times\vec{E}_{\lambda}\right)-\vec{E}_{\lambda}\times\left(Z^{-1}_{\lambda'}\hat{z}\times\vec{E}_{\lambda'}\right)\right)_z e^{i(k_\lambda+k_{\lambda'})z}\\
&\quad=\sum_{\lambda'}A^+_{\lambda'}\int_{A}\left(\frac{1}{Z_\lambda}\vec{E}_{\lambda'}\cdot\vec{E}_{\lambda}-\frac{1}{Z_{\lambda'}}\vec{E}_{\lambda}\cdot\vec{E}_{\lambda'}\right)e^{i(k_\lambda+k_{\lambda'})z}\\
&\quad=\sum_{\lambda'}A^+_{\lambda'}\delta_{\lambda\lambda'}\left(\frac{1}{Z_\lambda}-\frac{1}{Z'_\lambda}\right)e^{i(k_\lambda+k_{\lambda'})z}=0.
\end{aligned}$$

On the other hand, the contribution from S_- is

$$\begin{aligned}
&\sum_{\lambda'}A^-_{\lambda'}\int_{S_-}-\left(\vec{E}^-_{\lambda'}\times\vec{H}^+_{\lambda}-\vec{E}^+_{\lambda}\times\vec{H}^-_{\lambda'}\right)\cdot\hat{z}\\
&\quad=\sum_{\lambda'}A^-_{\lambda'}\int_{S_-}-\left(\vec{E}_{\lambda'}\times\vec{H}_{\lambda}+\vec{E}_{\lambda}\times\vec{H}_{\lambda'}\right)\cdot\hat{z}\,e^{i(k_\lambda-k_{\lambda'})z}\\
&\quad=-\sum_{\lambda'}A^-_{\lambda'}\frac{2}{Z_\lambda}\delta_{\lambda\lambda'}=-\frac{2}{Z_\lambda}A^-_{\lambda}
\end{aligned}$$

so

$$A^-_{\lambda}=-\frac{Z_\lambda}{2}\int_V\vec{J}\cdot\vec{E}^+_{\lambda}.$$

The same argument for the lower sign, as spelled out in the book, gives the equation with the superscript signs reversed, so both are

$$A_\lambda^\pm = -\frac{Z_\lambda}{2}\int_V \vec{J}\cdot\vec{E}_\lambda^\mp.$$

In addition to sources due to currents, we may have contributions due to obstacles or holes in the conducting boundaries. These can be treated as additional surface terms in Gauss' law (by excluding obstacles from the region of integration V), but this requires knowing the full fields at the surface of the obstacles or the missing parts of the waveguide conductor. This is treated in §9.5B, but we won't discuss it here.

So finally we are at the end of Chapter 8.