Physics 504, Lecture 3
Jan. 28, 2010

3.1 Attenuation for Circular Cylinder

We have seen that the TE and TM modes in a circular wave guide are determined by

$$\psi^{\text{TE}}_{mn} = J_m(x'_{mn}\rho/r)\cos m\phi, \qquad \psi^{\text{TM}}_{mn} = J_m(x_{mn}\rho/r)\cos m\phi,$$

where x_{mn} and x'_{mn} are the n'th zeros of $J_m(x)$ and $J'_m(x)$ respectively. The cutoff frequencies are given in terms of

$$\gamma^{\text{TE}}_{mn} = x'_{mn}/r, \qquad \gamma^{\text{TM}}_{mn} = x_{mn}/r.$$

To evaluate the attenuation coeffieients, we need $\int_A \psi^2$, and

$$\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 = r\int_0^{2\pi} d\phi\gamma^2 J'^2_m(\gamma r)\cos^2\phi = \pi r\gamma^2 J'^2_m(\gamma r)(1+\delta_{m0}),$$

which we need only for TM modes. For TE modes we need

$$\int_\Gamma |\psi|^2 = rJ^2_m(x'_{mn})\int_0^{2\pi}\cos^2 m\phi\, d\phi = \pi r J^2_m(x'_{mn})(1+\delta_{m0}),$$

$$\begin{aligned}\int_\Gamma |\hat{n}\times\nabla_t\psi|^2 &= r\int_0^{2\pi} d\phi\left(\frac{\partial\psi}{r\partial\phi}\right)^2 = \frac{1}{r}J^2_m(x'_{mn})\int_0^{2\pi}(m\sin m\phi)^2 \\ &= \frac{\pi}{r}J^2_m(x'_{mn})(1+\delta_{m0}),\end{aligned}$$

The only integral that requires more than table look-up is

$$\int_A \psi^2 = \int_0^r \rho d\rho J^2_m(\gamma\rho)\int_0^{2\pi} d\phi\cos^2(m\phi) = \pi\int_0^r \rho d\rho J^2_m(\gamma\rho)(1+\delta_{m0}).$$

The integral is connected to the orthnormalization properties of Bessel functions, and is[1]:

$$\int_0^1 \left[J_m\left(x_{mn}u\right)\right]^2 u du = \frac{1}{2}J^2_{m+1}(x_{mn})$$

[1] Arfken 11.50, problems 11.2.2 and 11.2.3 (3rd Ed.) or see `http://www.physics.rutgers.edu/grad/504/lects/bessel.pdf`.

$$\int_0^1 \left[J_m\left(x'_{mn}u\right)\right]^2 u du = \frac{1}{2}\left(1 - \frac{m^2}{(x'_{mn})^2}\right) J_m^2(x'_{mn})$$

Thus for the TM modes, we have

$$\frac{C}{A}\xi_{mn}^{\text{TM}} = \int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Big/ (\gamma_{mn}^{\text{TM}})^2 \int_A \psi^2 = \frac{\pi r J_m'^2(x_{mn})}{\frac{\pi r^2}{2} J_{m+1}^2(x_{mn})} = \frac{2}{r}\frac{J_m'^2(x_{mn})}{J_{m+1}^2(x_{mn})}$$

In fact, there is an identity (see footnote again) $J'_m(x) = \frac{m}{x}J_m(x) - J_{m+1}(x)$, which means, as $J_m(x_{mn}) = 0$, that $J'_m(x_{mn}) = -J_{m+1}(x_{mn})$, $\frac{C}{A}\xi_{mn}^{\text{TM}} = 2/r$, and

$$\beta_{mn}^{\text{TM}} = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{r\sigma\delta_\lambda}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}}$$

for all TM modes.

For the TE modes,

$$\begin{aligned}\frac{C}{A}\xi_{mn}^{\text{TE}} &= \int_\Gamma |\hat{n} \times \nabla_t\psi|^2 \Big/ (\gamma_{mn}^{\text{TE}})^2 \int_A \psi^2 = \frac{m^2 \pi J_m^2(x'_{mn})/r}{\pi(\gamma_{mn}^{\text{TE}})^2 r^2 \frac{1}{2}\left(1 - (m/x'_{mn})^2\right) J_m^2(x'_{mn})} \\ &= \frac{2m^2}{r(x_{mn}'^2 - m^2)}.\end{aligned}$$

$$\frac{C}{A}\zeta_{mn}^{\text{TE}} = \int_\Gamma |\psi|^2 \Big/ \int_A \psi^2 = \frac{\pi r J_m^2(x'_{mn})}{\frac{\pi}{2}\left(1 - (m/(x'_{mn})^2)\right) J_m^2(x'_{mn})} = \frac{2x_{mn}'^2}{r\left(x_{mn}'^2 - m^2\right)}.$$

So the attenuation coefficient is

$$\beta_{mn}^{\text{TE}} = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{r\sigma\delta_\lambda}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}}}\left[\frac{1}{(x_{mn}'^2 - m^2)} + \left(\frac{\omega_\lambda}{\omega}\right)^2\right].$$

For TM modes, $\omega_{mn}^{\text{TM}} = x_{mn}c/r$, where $c = 1/\sqrt{\mu\epsilon}$ is the speed of light, For copper, the resistivity is $\rho = \sigma^{-1} = 1.7 \times 10^{-8}\ \Omega \cdot \text{m}$, and we may take the permeability to be essentially μ_0. Also $\omega_\lambda = \gamma_\lambda c$. $\delta_\lambda = \sqrt{2/\mu_c\sigma\omega_\lambda}$ $\epsilon_0 = 8.854 \times 10^{-12}\ \text{C}^2/\text{N} \cdot \text{m}^2$, so

$$\sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda} = \sqrt{\frac{c\epsilon_0\gamma_\lambda}{2\sigma}} = 4.75{\times}10^{-6}\sqrt{\gamma_\lambda}\sqrt{\frac{\text{m}}{\text{s}}\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\Omega\text{m}} = 4.75{\times}10^{-6}\ \text{m}^{1/2}\cdot\sqrt{\frac{x_{mn}}{r}}.$$

The units combine to $\mathrm{m}^{1/2}$ as $1\ \Omega = 1\mathrm{V/A} = 1(\mathrm{J/C})/(\mathrm{C/s}) = \mathrm{Nms/C}^2$.

In comparison to the TM_{12} mode for a square of side a, we see that $\beta^{\mathrm{TM}} = \frac{a}{2r}\beta_{12}^{\square\mathrm{TM}}$. As the cutoff frequencies are $2.4048c/r$ and $\sqrt{5}\pi c/a$ respectively, we see that the comparable dimensions are $r = (2.4048/\sqrt{5}\pi)a = 0.342a$, much smaller, and then $a/2r = 1.46$, so the smaller pipe does have faster attenuation.

For TE modes, there is an extra factor of

$$\frac{1}{(x_{mn}'^2 - m^2)} + \left(\frac{\omega_\lambda}{\omega}\right)^2.$$

which for the lowest mode is $0.4185 + (\omega_\lambda/\omega)^2$ compared to $0.5 + (\omega_\lambda/\omega)^2$ for the square. But the cutoff frequencies are now $1.841c/r$ and $\sqrt{2}\pi c/a$, so comparable dimensions have $r = 1.841a/\sqrt{2}\pi = 0.414a$.

3.2 Resonant Cavities

We have considered wave guides uniformly extended in the z direction, infinite in both directions, and found that there are modes λ of propagation with arbitrary definite wavenumber k and frequency ω given by $\mu\epsilon\omega^2 = k^2 + \gamma_\lambda^2$. Thus for a particular λ and $\omega > \omega_\lambda$, there are two possible waves, a right-moving and a left-moving one. Superposition will then give us standing waves suitable to describe a resonant cavity made by placing flat conducting end-caps on the wave guide, say at $z = 0$ and $z = d$.

Thus quite generally each field will be a superposition of wave with $k = |k|$ and one with $k = -|k|$. For the TM case, the determining field is

$$E_z = \left(\psi^{(k)} e^{ikz} + \psi^{(-k)} e^{-ikz}\right) e^{-i\omega t},$$

In calculating the transverse field, the piece coming from $\psi^{(-k)}$ needs the minus in 8.33, so

$$\vec{E}_t = i\frac{k}{\gamma_\lambda^2}\left(\vec{\nabla}_t\psi^{(k)} e^{ikz} - \vec{\nabla}_t\psi^{(-k)} e^{-ikz}\right)$$

This is a field parallel to the conductor surface at $z = 0$ and $z = d$, so must vanish (or be very small) there for a perfect (good) conductor endcap. Vanishing of $\vec{E}_t$ at $z = 0$ implies $\psi^{(k)} = \psi^{(-k)}$, and then $\psi^{(k)}(2i \sin kd) = 0$

from $z = d$. As we don't want $\psi = 0$, we see that $k = p\pi/d$ with p an integer,

$$\left.\begin{aligned} E_z &= \cos\left(\frac{p\pi z}{d}\right)\,\psi(x,y) \\ \vec{E}_t &= -\frac{p\pi}{d\gamma_\lambda^2}\sin\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \\ \vec{H}_t &= i\frac{\epsilon\omega}{\gamma_\lambda^2}\cos\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \end{aligned}\right\} \qquad \begin{cases}\text{for TM modes} \\ \text{with } p\in\mathbb{Z}\end{cases}$$

where in using 8.26, we recall that k takes a minus sign for the part of the field flowing leftward.

For TE modes, the determining field is H_z, which must vanish at $z = 0$ and $z = d$ if the endcaps are perfect conductors and exclude magnetic fields, as $\hat{n}\cdot\vec{B}$ is continuous at the boundary. Thus

$$\left.\begin{aligned} H_z &= \sin\left(\frac{p\pi z}{d}\right)\,\psi(x,y) \\ \vec{H}_t &= \frac{p\pi}{d\gamma_\lambda^2}\cos\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \\ \vec{E}_t &= -i\frac{\omega\mu}{\gamma_\lambda^2}\sin\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \end{aligned}\right\} \qquad \begin{cases}\text{for TE modes} \\ \text{with } p\in\mathbb{Z}, p\neq 0.\end{cases}.$$

As for the waveguide, the values of γ_λ depend on the mode (TE or TM) and the cross section, which often means two indices. For example, for a circular guide, there is an angular index m and another index n specifying which root of J_m (for TM) or of $dJ(x)/dx$ (for TE). With x_{mn} the n'th zero of $J_m(x)$ (not counting $x = 0$) and x'_{mn} the n'th zero of $\frac{dJ}{dx}(x)$, we have $\gamma_{mn} = x_{mn}/R$ (TM modes) or $\gamma_{mn} = x'_{mn}/R$ (TE modes). As $\mu\epsilon\omega^2 = k^2+\gamma^2$ we have

$$\begin{aligned} \omega_{mnp} &= \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x_{mn}^2}{R^2}+\frac{p^2\pi^2}{d^2}} \quad \text{with } p\geq 0 \text{ for TM modes,} \\ \omega_{mnp} &= \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x'^2_{mn}}{R^2}+\frac{p^2\pi^2}{d^2}} \quad \text{with } p> 0 \text{ for TE modes.} \end{aligned}$$

The lowest TM mode is thus $\omega_{010} = x_{01}/\sqrt{\mu\epsilon}\,R = 2.405/\sqrt{\mu\epsilon}\,R$, which is independent of the length d of the cavity. As p cannot be zero for TE modes, (or the determining field $H_z = 0$) the lowest TE mode is

$$\omega_{111} = \frac{1.841}{\sqrt{\mu\epsilon}R}\sqrt{1+2.912\frac{R^2}{d^2}},$$

($\pi/1.841 = 2.912$). This mode has the advantage that its frequency can be tuned by moving a piston back and forth, changing d.

This calculation has assumed no power losses, but of course a real cavity will generally have walls of finite conductivity, and power will be lost as we discussed earlier in the walls, not only along the z direction but also in the endcaps. Again the power lost will be proportional to the energy stored in the cavity. Let Q be the 2π times the energy stored U divided by the energy lost in one cycle ΔU (in time $dt = 2\pi/\omega$), so $Q = 2\pi\frac{U}{\Delta U}$. Assuming $Q \gg 1$, the energy loss per cycle will be small compared to U, with $\Delta U \approx -\frac{2\pi}{\omega}\frac{dU}{dt}$, and the energy will decay exponentially, with

$$U(t) = U(0)e^{-\omega t/Q}, \quad Q = \omega U/|dU/dt|.$$

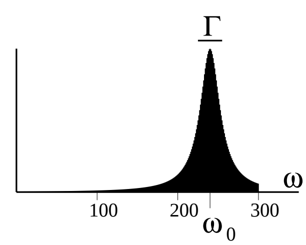
This means that if at time zero something excites an electromagnetic field in the cavity, the fields will have a time dependence[2]

$$E(t) = E_0 e^{-i\omega_0(1-i/2Q)t}\Theta(t),$$

where $\Theta(t) = 1$ for $t \geq 0$ and zero earlier. Thus the frequency response to what is essentially a delta-function excitation (and therefore equal for all frequencies) is

$$\begin{aligned} E(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt = \frac{1}{\sqrt{2\pi}}E_0\int_0^{\infty} e^{i(\omega-\omega_0-i\Gamma/2)t}dt \\ &= \frac{iE_0}{\sqrt{2\pi}}\frac{1}{\omega-\omega_0-i\Gamma/2}, \end{aligned}$$

where $\Gamma := \omega_0/Q$. This response determines how the cavity will respond to excitations of any frequency, with the energy absorbed proportional to



$$|E(\omega)|^2 \propto \frac{1}{(\omega-\omega_0)^2 + \Gamma^2/4}.$$

This resonance shape is the form of response the simplest resonant structures have in response to a

[2]My ω_0 is what Jackson calls $\omega_0 + \Delta\omega$, with his ω_0 the resonant frequency of the undamped cavity. The change in resonant frequency due to damping is generally small, a fractional change of the order $1/Q$, and I will ignore that effect.

stimulus of frequency ω, as for example the ratio of the energy of damped harmonic oscillator to the driving force, or the resonant absorption of light by an atomic transition. In nuclear physics this is called the Breit-Wigner amplitude. Γ, mistakenly called the half-width, is actually the full width of the region with a response at least half the maximum value, which is $\omega \in [\omega_0 - \Gamma/2, \omega_0 + \Gamma/2]$.

The value of Γ for a resonant cavity can be calculated as for the attenuation of a waveguide. That is, we compare the power lost in the walls to the energy in the electromagnetic fields in the cavity. This is done in Jackson, pp 373-374, based on the same tools as used in calculating the attenuation of the waveguide, but I will skip it.

4 Earth and Ionosphere

An interesting resonant cavity is formed by the surface of the Earth and the ionosphere, a layer of ionized gas starting about 100 km up, which provides sufficient conductivity to reflect radio waves. But this cavity is not a cylinder with endcaps, but clearly calls out for spherical coordinates. As you also need an introduction to other curvilinear coordinate systems to do your homework, and as you have told me you have not learned about these, let us digress to discuss curvilinear coordinates in general, orthogonal coordinates more particularly, and finally spherical coordinates.