Physics 504, Lecture 2 part 2
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1 Energy Flow, Density and Attenuation

We have seen that there are discrete modes λ for electromagnetic waves with $\vec{E}, \vec{B} \propto e^{ikz-i\omega t}$ corresponding, for real ω and k, to waves travelling in the z direction, with a dispersion relation $k^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2$, with discrete values of γ_λ for TE or TM modes, or $\gamma = 0$ for TEM modes. This relation for TE and TM modes are of the same form as for the high-frequency behavior of dielectrics, (J 7.61) with $\gamma/\sqrt{\mu\epsilon}$ playing the roll of the plasma frequency. The phase velocity is

$$v_p = \omega/k = \frac{1}{\sqrt{\mu\epsilon}} \frac{1}{\sqrt{1 - \left(\frac{\omega_\lambda}{\omega}\right)^2}} > \frac{1}{\sqrt{\mu\epsilon}},$$

greater than the velocity in an infinite medium ($\mathbb{R}^3$), as for all travelling waves we have $\omega > \omega_\lambda$.

On the other hand, as $k^2 - \mu\epsilon\omega^2 =$ constant, we have $kdk = \mu\epsilon\omega d\omega$, so the group velocity

$$v_g = \frac{d\omega}{dk} = \frac{1}{\mu\epsilon}\frac{k}{\omega} = \frac{1}{\mu\epsilon}\frac{1}{v_p} < \frac{1}{\sqrt{\mu\epsilon}},$$

less than the infinite medium velocity.

1.1 Energy flow and density

These calculations assumed the walls had infinite conductivity, but generally the walls are good but not perfect conductors,. Then there will be eddy currents and energy loss in the walls, and a right-moving wave will be attenuated in the z direction, so k will develop a small positive imaginary part. We may estimate this attenuation constant by comparing the energy travelling past z to the energy per unit length lost at z. The energy flux is given by the z component of the Poynting vector, so the power transmitted down the pipe is $P = \int_A \hat{z} \cdot \vec{S}$.

Up to now we have worked with quantities linear in $\vec{E}$ or $\vec{H}$ so the proviso that the real fields are just the real parts of our complex fields could be swept under the rug. But the Poynting vector $\vec{S} = \vec{E}_{\rm phys} \times \vec{H}_{\rm phys}$ is quadratic. Our wave propagating in the z direction is actually

$$\vec{E}_{\rm phys}(x,y,z,t) = \frac{1}{2}\left(\vec{E}(x,y,k,\omega)e^{ikz-i\omega t} + \vec{E}^*(x,y,k,\omega)e^{-ikz+i\omega t}\right).$$

So when we talk in terms of a wave with a single k and ω,

$$\vec{E}_{\rm phys}(x,y,z,t) = \text{ Re } \vec{E}(x,y,z,t) = \text{ Re } \left(\vec{E}(x,y,k,\omega)e^{ikz-i\omega t}\right),$$

we have

$$\begin{aligned}
\vec{S}_{\rm phys} &= \vec{E}_{\rm phys} \times \vec{H}_{\rm phys} \\
&= \frac{1}{4}\Bigg(\left(\vec{E}(x,y,k,\omega)e^{ikz-i\omega t} + \vec{E}^*(x,y,k,\omega)e^{-ikz+i\omega t}\right) \times \\
&\qquad\qquad \left(\vec{H}(x,y,k,\omega)e^{ikz-i\omega t} + \vec{H}^*(x,y,k,\omega)e^{-ikz+i\omega t}\right)\Bigg) \\
&= \frac{1}{4}\Bigg(\vec{E}(x,y,k,\omega) \times \vec{H}(x,y,k,\omega)e^{2ikz-2i\omega t} \\
&\quad +\vec{E}^*(x,y,k,\omega) \times \vec{H}(x,y,k,\omega) + \vec{E}(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega) \\
&\quad +\vec{E}^*(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega)e^{-2ikz+2i\omega t}\Bigg)
\end{aligned}$$

The first and last term are rapidly oscillating, so if we are interested in the average of $\vec{S}$, we have

$$\begin{aligned}
\langle\vec{S}\rangle &= \frac{1}{4}\left(\vec{E}^*(x,y,k,\omega) \times \vec{H}(x,y,k,\omega) + \vec{E}(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega)\right) \\
&= \frac{1}{2}\text{ Re }\left(\vec{E}(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega)\right)
\end{aligned}$$

The power transmitted down the waveguide is the integral of the z component of this, so only the transverse components of $\vec{E}$ and $\vec{H}$ are needed, and these are given by

$$\begin{array}{llll}
\text{TM:} & E_z = \psi, \quad \vec{E}_t = i\dfrac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi, & \vec{H}_t = i\dfrac{\epsilon\omega}{\gamma_\lambda^2}\hat{z} \times \vec{\nabla}_t\psi, & \psi|_S = 0 \\
\text{TE:} & H_z = \psi, \quad \vec{H}_t = i\dfrac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi, & \vec{E}_t = -i\dfrac{\mu\omega}{\gamma_\lambda^2}\hat{z} \times \vec{\nabla}_t\psi, & \dfrac{\partial\psi}{\partial n}|_S = 0
\end{array}$$

with $(\nabla_t^2 + \gamma_\lambda^2)\psi = 0$. Thus the average transmitted power is

$$P = \hat{z}\cdot \ \mathrm{Re} \int_A S = \frac{\omega k}{2\gamma_\lambda^4}\int_A |\vec{\nabla}_t\psi|^2 \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

The integral

$$\int_A |\vec{\nabla}_t\psi|^2 = \oint_S \psi^* \frac{\partial\psi}{\partial n} - \int_A \psi^*\nabla_t^2\psi = 0 + \gamma_\lambda^2 \int_A \psi^*\psi.$$

With $\omega_\lambda := \gamma_\lambda/\sqrt{\mu\epsilon}$, $k = \omega\sqrt{\mu\epsilon}\sqrt{1-\omega_\lambda^2/\omega^2}$, this is

$$P = \frac{1}{2\sqrt{\mu\epsilon}}\left(\frac{\omega}{\omega_\lambda}\right)^2 \sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}\int_A \psi^*\psi \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

We might also calculate the energy per unit length in the waveguide, $U = \int_A u = \frac{1}{2}\int_A \left(\vec{E}_{\text{phys}}\cdot\vec{D}_{\text{phys}} + \vec{B}_{\text{phys}}\cdot\vec{H}_{\text{phys}}\right)$, so

$$\langle U\rangle = \frac{1}{4}\int_A \epsilon|\vec{E}|^2 + \mu|\vec{H}|^2$$

Here the longitudinal ($\hat{z}$) components enter as well as the transverse ones. For the TM mode, we have

$$\begin{aligned} |\vec{H}|^2 = |\vec{H}_t|^2 &= \left(\frac{\epsilon\omega}{\gamma_\lambda^2}\right)^2 \left|\vec{\nabla}_t\psi\right|^2 \\ |\vec{E}|^2 = |\vec{E}_t|^2 + |\vec{E}_z|^2 &= \left(\frac{k}{\gamma_\lambda^2}\right)^2 \left|\vec{\nabla}_t\psi\right|^2 + |E_z|^2 \end{aligned}$$

As $E_z = \psi$,

$$\begin{aligned} \langle U\rangle &= \frac{1}{4}\left[\epsilon\left(\frac{k^2}{\gamma_\lambda^4}\int_A |\vec{\nabla}_t\psi|^2 + \int_A |\psi|^2\right) + \mu\left(\frac{\epsilon\omega}{\gamma_\lambda^2}\right)^2 \int_A |\vec{\nabla}_t\psi|^2\right] \\ &= \frac{\epsilon}{2}\frac{\mu\epsilon\omega^2}{\gamma_\lambda^2}\int_A |\psi|^2 = \frac{\epsilon}{2}\frac{\omega^2}{\omega_\lambda^2}\int_A |\psi|^2 \qquad \text{(TM mode).} \end{aligned}$$

Similarly for the TE mode, we need $\vec{H}_t = ik\gamma_\lambda^{-2}\vec{\nabla}_t\psi$ and $H_z = \psi$

$$\int_A |\vec{H}|^2 = \frac{k^2}{\gamma_\lambda^4}\int_A \left|\vec{\nabla}\psi\right|^2 + \int_A |\psi|^2 = \left(\frac{k^2}{\gamma_\lambda^2}+1\right)\int_A |\psi|^2 = \frac{\omega^2}{\omega_\lambda^2}\int_A |\psi|^2$$

while

$$\int_A |\vec{E}|^2 = \frac{\mu^2\omega^2}{\gamma_\lambda^4} \int_A |\vec{\nabla}\psi|^2 = \frac{\mu}{\epsilon}\frac{\omega^2}{\omega_\lambda^2} \int_A |\psi|^2,$$

so

$$\langle U\rangle = \frac{\mu}{2}\frac{\omega^2}{\omega_\lambda^2} \int_A |\psi|^2 \qquad \text{(TE mode)}.$$

Note that in either case,

$$\frac{\langle P\rangle}{\langle U\rangle} = \frac{1}{\sqrt{\epsilon\mu}}\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} = v_g.$$

As we might expect, the energy flux is the energy density times the group velocity.

1.2 Attenuation

We now turn to the major effect of the finiteness of the conductivity σ in the walls of the waveguide. We saw in section 8.1 that the power loss per unit area of interface is given by

$$\frac{dP_{\rm loss}}{dA} = \frac{1}{2\delta\sigma}\left|\vec{H}_\parallel\right|^2,$$

where $\delta = \sqrt{2/\mu_c\sigma\omega}$ is the skin depth. This gives a loss of power per unit length proportional to the square of the fields and therefore to the power, so the power transmitted along the waveguide will fall off exponentially. This can be described by giving k a small imaginary part, $\operatorname{Im} k = \beta_\lambda$, giving a factor $e^{-\beta_\lambda z}$ to each of the fields, and $e^{-2\beta_\lambda z}$ to the power flow. Thus

$$\frac{dP}{dz} = -2\beta_\lambda P(z) = -\frac{1}{2\sigma\delta}\oint_\Gamma \left|\hat{n}\times\vec{H}\right|^2 d\ell.$$

For the TM mode,

$$\hat{n}\times\vec{H} = \hat{n}\times\vec{H}_t = \frac{i\epsilon\omega}{\gamma_\lambda^2}\hat{n}\times(\hat{z}\times\vec{\nabla}_t\psi) = \frac{i\epsilon\omega}{\gamma_\lambda^2}\left(\hat{n}\cdot\vec{\nabla}_t\psi\right)\hat{z}$$

so

$$\beta_\lambda = \frac{1}{4\sigma\delta}\left(\frac{\epsilon\omega}{\gamma_\lambda^2}\right)^2 \int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Big/ \frac{\omega k\epsilon}{2\gamma_\lambda^4}\int_A \left|\vec{\nabla}\psi\right|^2 = \frac{\omega\epsilon}{2k\sigma\delta}\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Big/ \int_A \left|\vec{\nabla}\psi\right|^2$$

Though we don't know the ratio of the integrals, they differ only in that numerator takes only the square of one of the two components of $\vec{\nabla}\psi$, and integrates it over the circumference rather than the area. Furthermore the ratio of these integrals depends only on the mode λ and not on the frequency ω of the transmitted wave. Let us write

$$\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Big/ \int_A \left|\vec{\nabla}\psi\right|^2 = \frac{C}{A}\xi_\lambda,$$

where C and A are the circumference and the area of the wave guide's cross section, and ξ_λ is a dimensionless number expected to be of order-of-magnitude 1. Let us display the frequency dependance explicitly by writing $\delta = \delta_\lambda\sqrt{\omega_\lambda/\omega}$, $k = \omega\sqrt{\mu\epsilon}\sqrt{1-\omega_\lambda^2/\omega^2}$. Thus

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{\sigma\delta_\lambda}\,\frac{C}{2A}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\dfrac{\omega_\lambda^2}{\omega^2}}}\,\xi_\lambda.$$

For the TE mode, $\hat{n}\times\vec{H} = \hat{n}\times\vec{H}_t + \hat{n}\times\hat{z}H_z$ so

$$\left|\hat{n}\times\vec{H}\right|^2 = \left|\hat{n}\times\vec{H}_t\right|^2 + |H_z|^2 = \left(\frac{k}{\gamma_\lambda^2}\right)^2\left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 + |\psi|^2.$$

Again let us write

$$\int_\Gamma \left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 \Big/ \int_A \left|\vec{\nabla}\psi\right|^2 = \frac{C}{A}\xi_\lambda, \qquad \int_\Gamma |\psi|^2 \Big/ \int_A |\psi|^2 = \frac{C}{A}\zeta_\lambda.$$

where ζ_λ is another dimensionless number of order one, and ξ_λ is somewhat differently defined from the TM case, but still of order one. Then

$$\int_\Gamma \left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 \Big/ \int_A |\psi|^2 = \gamma_\lambda^2\frac{C}{A}\xi_\lambda.$$

The attenuation coefficient is

$$\beta_\lambda \;=\; \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{\sigma\delta_\lambda}\,\frac{C}{2A}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\dfrac{\omega_\lambda^2}{\omega^2}}}\left[\xi_\lambda\left(1-\frac{\omega_\lambda^2}{\omega^2}\right) + \zeta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right]$$

Then for both modes we can write the attenuation coefficient as

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}} \frac{1}{\sigma\delta_\lambda} \frac{C}{2A} \frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}} \left[\xi_\lambda + \eta_\lambda \left(\frac{\omega_\lambda}{\omega}\right)^2\right],$$

where $\eta_\lambda = \zeta_\lambda - \xi_\lambda$ for the TE mode. Note that for the TM mode $\eta_\lambda = 0$.

The attenuation coefficient calculated in this approximation diverges as $\omega \to \omega_\lambda$, and is it is proportional to $\sqrt{\omega}$ for large ω, so it has a minimum for some small multiple of ω_λ, $\sqrt{3}$ for the TM modes, but dependent on η_λ/ξ_λ, and hence the shape of the waveguide, for TE modes.

We will skip section 8.6.