Physics 504, Lectures 1-2
Jan. 21 - 25, 2010

1 Lecture 1

Prof. Thomas covered Chapters 1-7 in Jackson, pretty completely, as I understand it. We will cover the remaining chapters, 8-16, but probably we will leave out a greater fraction of the topics.

The basics of classical electrodynamics are very simple: Maxwell's Equations and the Lorentz Force. The applications, however, are many, important, and often quite involved. After a brief review of Maxwell's equations, we will begin with a discussion of electromagnetic fields within a region bounded by materials which tend to confine them — waveguides, cavities, and optical fibers. Then we will discuss sources of such fields, such as antennas, and the radiation that results. This will be followed by scattering and diffraction.

After these rather applied subjects, we will turn to some formal discussions, in particular special relativity and the relativistic forms of expressing electromagnetism. I should remind you that it was by considering electromagnetic effects that Einstein was led to special relativity. Then we will return to applications, now focussing on possibly relativistic particles.

First, let's review the basic equations:

1.1 Maxwell's Equations

1.1.1 Fundamental

In terms of total charge density $\rho_{\rm all}$ and total current $\vec{J}_{\rm all}$, which include polarization charges and magnetization, Maxwell's Equations are

$$\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \frac{1}{\epsilon_0}\rho_{\rm all} &(1)\\ \vec{\nabla}\cdot\vec{B} &= 0 &(2)\\ \vec{\nabla}\times\vec{B}-\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} &= \mu_0\vec{J}_{\rm all} &(3)\\ \vec{\nabla}\times\vec{E}+\frac{\partial\vec{B}}{\partial t} &= 0 &(4)\end{aligned}$$

and the Lorentz force is

$$\vec{F} = q(\vec{E}+\vec{v}\times\vec{B}). \tag{5}$$

$\vec{E}(\vec{x},t)$ is the **electric field**, $\vec{B}(\vec{x},t)$ is, unfortuately, called the **magnetic induction**.

Equations 1 and 2 are Gauss's laws for the electric and magnetic fields respectively, equation 3 is Ampère's law with Maxwell's addition of the displacement current, and equation 4 is Faraday's law.

1.1.2 With "Ponderable Media"

The charge and current densities $\rho_{\rm all}$ and $\vec{J}_{\rm all}$ include all charges and currents, both "free" and those induced in materials by the electromagnetic fields. It is often useful to separate out the field due to these, and define the **electric polarization** $\vec{P}$ and the **electric displacement** $\vec{D}$, with

$$\vec{D} = \epsilon_0\vec{E}+\vec{P}, \tag{6}$$

where $\vec{D}$ has as its source only the "free charge" ρ:

$$\vec{\nabla}\cdot\vec{D} = \rho. \tag{7}$$

The polarization is given by the electric dipole moments of the molecules of the material. The induced charge density is then $-\vec{\nabla}\cdot\vec{P}$.

Similarly, the material will respond to electromagnetic fields with microscopic currents which lead to a **magnetization** $\vec{M}(\vec{x})$ in terms of which the induced or **effective current density** is $\vec{\nabla}\times\vec{M}$, and we define the **magnetic field** $\vec{H}(\vec{x},t)$ by

$$\vec{H} = \frac{1}{\mu_0}\vec{B}-\vec{M}, \tag{8}$$

with

$$\vec{\nabla}\times\vec{H}-\frac{\partial\vec{D}}{\partial t} = \vec{J}, \tag{9}$$

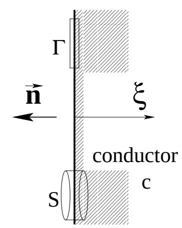
where $\vec{J}(\vec{x},t)$ refers only to the current of free charges.

Although the name "magnetic field", historically speaking, refers to H and not B, it is often used, when describing situations without ponderable media, to refer to B, which is the fundamental field, along with E, that describes electromagnetism.

1.2 Interface between conductor and non-conductor

Consider a smooth interface between a good conductor c and a nonconducting region. There may be a surface charge Σ. If the conductor is perfect, there will be no electric field inside, and for time-varying fields, there will be a surface current or eddy current, which prevents H from entering the conductor.

From the boundary conditions, and the finiteness of $\vec{B}$, Faraday gives us $\int \vec{E}\cdot d\ell = 0$ around a little loop Γ. This tells us that the components of $\vec{E}$ parallel to the surface are continuous, and are small or zero even just outside the conductor, though the normal component is not surpressed. For the magnetic field, considering the little pillbox S shows the normal component of B is continuous, and therefore surpressed, but components parallel to the surface can change quickly or discontinuously due to surface currents in a good or perfect conductor.



Consider a good but not perfect conductor, with a large but finite conductivity σ, with $\vec{J} = \sigma\vec{E}$. Then $\vec{J}$ is not a delta function of depth ξ, but rather is spread out in a layer roughly the "skin depth" deep, with H varying continuously across the boundary but rapidly across the skin depth, going to zero well within the conductor. Then the current density will be large, and in Ampère's law J will dominate over the displacement current, so we can write

$$\vec{\nabla}\times\vec{H}_c = \vec{J} + \frac{\partial\vec{D}}{\partial t} \approx \sigma\vec{E},$$

while Faraday gives

$$\vec{\nabla}\times\vec{E}_c = -\frac{\partial\vec{B}}{\partial t} = i\omega\mu_c H_c,$$

assuming a time dependence proportional to $e^{-i\omega t}$. The most rapid variation is with depth: ignoring other components

$$\vec{\nabla} = -\hat{n}\frac{\partial}{\partial\xi}, \qquad \vec{E}_c = \frac{1}{\sigma}\vec{J} = -\frac{1}{\sigma}\hat{n}\times\frac{\partial\vec{H}_c}{\partial\xi}, \qquad \vec{H}_c = \frac{i}{\omega\mu_c}\hat{n}\times\frac{\partial\vec{E}_c}{\partial\xi}, \tag{10}$$

so $\hat{n}\cdot\vec{H}_c = 0$ and

$$\begin{aligned}\hat{n}\times\vec{H}_c &= \frac{i}{\omega\mu_c}\hat{n}\times\left(\hat{n}\times\frac{\partial\vec{E}_c}{\partial\xi}\right) = -\frac{i}{\sigma\omega\mu_c}\hat{n}\times\left(\hat{n}\times\left[\hat{n}\times\frac{\partial^2\vec{H}_c}{\partial\xi^2}\right]\right)\end{aligned}$$

$$= \frac{i}{\sigma\omega\mu_c}\frac{\partial^2}{\partial\xi^2}\left(\hat{n}\times\vec{H}_c\right).$$

Defining the skin depth by

$$\delta = \sqrt{\frac{2}{\mu_c\omega\sigma}}$$

we have $(\partial^2/\partial\xi^2 + 2i/\delta^2)\hat{n}\times\vec{H}_c = 0$, or

$$\vec{H}_c = \vec{H}_\parallel e^{-\xi/\delta}e^{i\xi/\delta},$$

where $\vec{H}_\parallel$ is the tangential magnetic field at the surface outside the conductor.

To estimate the power loss in the skin, we will need the electric field from (10),

$$\vec{E}_c = -\frac{1}{\sigma}\hat{n}\times\frac{\partial\vec{H}_c}{\partial\xi} = \sqrt{\frac{\mu_c\omega}{2\sigma}}(1-i)\hat{n}\times\vec{H}_\parallel e^{-\xi/\delta}e^{i\xi/\delta},$$

which means, by continuity, that just outside the conductor

$$\vec{E}_\parallel = \sqrt{\frac{\mu_c\omega}{2\sigma}}(1-i)\hat{n}\times\vec{H}_\parallel.$$

There are two ways to calculate the power. First, the flow of energy through the surface is given by the Poynting vector $\vec{S} = \vec{E}\times\vec{H}$. Because we are using complex fields E and $H \propto e^{-i\omega t}$, of which only the real parts are physical, we need $\langle\vec{S}\rangle = \frac{1}{2}\,\mathrm{Re}\,\vec{E}\times\vec{H}^*$. So the power loss per unit area is

$$\begin{aligned}\frac{dP_{\text{loss}}}{dA} &= -\hat{n}\cdot\langle\vec{S}\rangle = -\frac{1}{2}\sqrt{\frac{\mu_c\omega}{2\sigma}}\hat{n}\cdot\ \mathrm{Re}\ \left[(1-i)(\hat{n}\times\vec{H}_\parallel)\times\vec{H}_\parallel^*\right] \\ &= \frac{\mu_c\omega\delta}{4}|\vec{H}_\parallel|^2 = \frac{1}{2\sigma\delta}|\vec{H}_\parallel|^2\end{aligned}$$

The other way is to ask about the ohmic losses, with power lost per unit volume of $\frac{1}{2}\vec{J}\cdot\vec{E}^* = |\vec{J}|^2/2\sigma$. As $|\vec{J}| = \sigma\vec{E}_c = \frac{\sqrt{2}}{\delta}|\vec{H}_\parallel|e^{-\xi/\delta}$, the power loss per unit area is

$$\frac{dP_{\text{loss}}}{dA} = \frac{1}{\delta^2\sigma}|\vec{H}_\parallel|^2\int_0^\infty d\xi\, e^{-2\xi/\delta} = \frac{1}{2\delta\sigma}|\vec{H}_\parallel|^2.$$

We can also express this in terms of the surface current, where we mean the total current near the surface,

$$\vec{K}_{\text{eff}} = \int_0^\infty d\xi\,\vec{J}(\xi) = \frac{1}{\delta}\hat{n}\times\vec{H}_\parallel\int_0^\infty d\xi\,(1-i)e^{-\xi(1-i)/\delta} = \hat{n}\times\vec{H}_\parallel.$$

Thus

$$\frac{dP_{\rm loss}}{dA} = \frac{1}{2\sigma\delta}|\vec{K}_{\rm eff}|^2.$$

Thus we may view $1/\sigma\delta$ as the surface resistance, or the ratio $\vec{E}_\parallel/\vec{K}_{\rm eff} = (1-i)/\sigma\delta$ as the surface impediance Z.

1.3 Waveguides

As our situation involves time-independent boundary conditions and linear equations, we can use a fourier transform in time, with

$$\begin{aligned}\vec{E}(\vec{x},t) &= \vec{E}(x,y,z)\; e^{-i\omega t}\\ \vec{B}(\vec{x},t) &= \vec{B}(x,y,z)\; e^{-i\omega t}\end{aligned}$$

with the understanding that the physical fields are the real part of these expressions, and of course we could have superpositions of different frequencies, but these don't interact.

In the interior $\rho = 0$, $\vec{J} = 0$, so

$$\vec\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B}, \qquad \vec\nabla\cdot\vec{E} = 0, \qquad \vec\nabla\cdot\vec{B} = 0,$$

$$\vec\nabla\times\vec{B} = \mu\vec\nabla\times\vec{H} = \mu\frac{\partial\vec{D}}{\partial t} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} = -i\omega\mu\epsilon\vec{E}.$$

Then

$$\nabla^2\vec{E} = -\vec\nabla\times(\vec\nabla\times\vec{E}) + \vec\nabla\left(\vec\nabla\cdot\vec{E}\right) = -\vec\nabla\times(i\omega\vec{B}) = -\omega^2\mu\epsilon\vec{E}.$$

and similarly

$$\left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{B} = 0. \tag{11}$$

Let us assume our problem involves a cylinder of arbitrary cross-section, but uniform in z (though possibly only on an interval in z, possibly capped at the ends). Then we can also fourier transform in z,

$$\vec{E}(x,y,z,t) = \vec{E}(x,y)e^{ikz-i\omega t}, \qquad \vec{B}(x,y,z,t) = \vec{B}(x,y)e^{ikz-i\omega t},$$

where k could take either sign, and we might take a superposition if we need to. Then the Helmholtz equation (11) for $\vec{B}$ and $\vec{E}$ give

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right]\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0, \quad \nabla_t^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Break down the vectors into transverse and longitudinal parts:

$$\vec{E} = E_z\hat{z} + \vec{E}_t, \quad \vec{B} = B_z\hat{z} + \vec{B}_t, \quad \text{with } \vec{E}_t\perp\hat{z},\ \vec{B}_t\perp\hat{z}.$$

Now

$$\begin{aligned}(\vec\nabla\times\vec{E})_z &= (\vec\nabla_t\times\vec{E}_t)_z = i\omega B_z, &(12)\\ (\vec\nabla\times\vec{E})_\perp &= \hat{z}\times\frac{\partial\vec{E}_t}{\partial z} - \hat{z}\times\nabla_t E_z = i\omega\vec{B}_t. &(13)\end{aligned}$$

For any vector $\vec{V}$, $\hat{z}\times(\hat{z}\times\vec{V}) = -\vec{V} + \hat{z}(\hat{z}\cdot V)$, so for a transverse vector $\hat{z}\times(\hat{z}\times\vec{V}_t) = -\vec{V}_t$. Taking $\hat{z}\times$ Eq. (13) gives

$$\frac{\partial\vec{E}_t}{\partial z} - \vec\nabla_t E_z = -i\omega\hat{z}\times\vec{B}_t. \tag{14}$$

The same decomposition of $\vec\nabla\times\vec{B} = -i\omega\mu\epsilon\vec{E}$ gives

$$\begin{aligned}\left(\vec\nabla_t\times\vec{B}_t\right)_z &= -i\omega\mu\epsilon E_z &(15)\\ \frac{\partial\vec{B}_t}{\partial z} - \vec\nabla_t B_z &= i\omega\mu\epsilon\hat{z}\times\vec{E}_t. &(16)\end{aligned}$$

Of course the divergencelessness of $\vec{E}$ and $\vec{B}$ give

$$\vec\nabla_t\cdot\vec{E}_t + \frac{\partial E_z}{\partial z} = 0, \qquad \vec\nabla_t\cdot\vec{B}_t + \frac{\partial B_z}{\partial z} = 0.$$

Making use of the fourier transform in z, we have

$$\begin{aligned}ik\vec{E}_t + i\omega\hat{z}\times\vec{B}_t &= \vec\nabla_t E_z &(17)\\ ik\vec{B}_t - i\omega\mu\epsilon\hat{z}\times\vec{E}_t &= \vec\nabla_t B_z &(18)\end{aligned}$$

Solving 18 for $\vec{B}_t$ and plugging into 17, and then the reverse for $\vec{E}_t$, gives

$$\begin{aligned}E_t &= i\frac{k\vec\nabla_t E_z - \omega\hat{z}\times\vec\nabla_t B_z}{\omega^2\mu\epsilon - k^2} &(19)\\ B_t &= i\frac{k\vec\nabla_t B_z + \omega\mu\epsilon\hat{z}\times\vec\nabla_t E_z}{\omega^2\mu\epsilon - k^2} &(20)\end{aligned}$$

Thus E_z and B_z determine the rest, unless $k^2 = k_0^2 := \mu\epsilon\omega^2$, in which case both E_z and B_z are zero. Then there are no longitudinal fields, and we call this a transverse electromagnetic (TEM) wave. It travels in the z direction at the speed $1/\sqrt{\mu\epsilon}$ which we would have for a plane wave in an infinite medium, and with the wave number $k = k_0 := \omega\sqrt{\mu\epsilon}$ that the wave would have in an infinite medium. These TEM fields satisfy $\vec{\nabla}_t \cdot \vec{E}_t = 0$, $\vec{\nabla}_t \times \vec{E}_t = i\omega B_z = 0$, and therefore $\vec{E}_t = -\vec{\nabla}_t\Phi$ for some (not necessarily singlevalued) function Φ on the cross section, with $\nabla^2\Phi = 0$. As $\vec{E}_\parallel = 0$ at the boundary, each boundary is an equipotential of Φ, and if the cross section is simply connected, the only solution is $\Phi =$ constant, $\vec{E}_t = 0$. Thus there can be no TEM wave on a simply connected cylinder, but the TEM is the principal wave on a coaxial cable (which has an inner and an outer conductor, with different Φ, or for parallel wires, as in an old 300 Ω television cable. Note that if μ and ϵ are nondispersive, so is the TEM wave, with no cutoff on the transmission frequency or wavelength.

For perfectly conducting waveguides we saw that at the boundary $\vec{n}\times\vec{E} = 0$, $\vec{n}\cdot\vec{B} = 0$. This means $E_z = 0$ and $\vec{E}_t \parallel \vec{n}$ at the boundary. From $\vec{n}\cdot(\partial\vec{B}_z/\partial z - i\mu\epsilon\omega\hat{z}\times\vec{E}_t - \vec{\nabla}_t B_z) = 0$, the first two terms vanish, the second because $\vec{E}_t \parallel \hat{n}$ at the boundary, so $\partial B_z/\partial n|_S = 0$. Thus E_z satisfies a Dirichlet zero condition and B_z satisfies a Neumann zero condition boundary conditions in two dimensions. For simply connected cross section, there are in general no nonzero solutions, except for certain discrete values of the constant $\mu\epsilon\omega^2 - k^2$, and the allowed values will, in general, be different for the two possibilities. So in general if there is a solution for one condition, say $E_z = 0$ on the boundary, we will have $B_z \equiv 0$, the magnetic field is purely transverse, and we call this a transverse magnetic (TM) mode. For the other condition, $\vec{n}\cdot\vec{\nabla}_t B_z = 0$ on the boundary, we have $E_z \equiv 0$, $\vec{E}$ is purely transverse, and this is called a transverse electric (TE) mode.

1.4 Waveguide impediance, modes, and cutoff frequencies

Note that for a TM mode with vanishing B_z, (19) and (20) give

$$\text{TM:}\qquad (k_0^2 - k^2)\vec{E}_t = ik\vec{\nabla}_t E_z, \qquad (k_0^2 - k^2)\vec{B}_t = i\mu\epsilon\omega\hat{z}\times\vec{\nabla}_t E_z$$

or $\vec{H}_t = \epsilon\omega k^{-1}\hat{z}\times\vec{E}_t$, while for a TE mode with vanishing E_z

$$\text{TE:}\qquad (k_0^2 - k^2)\vec{E}_t = -i\omega\hat{z}\times\vec{\nabla}_t B_z, \qquad (k_0^2 - k^2)\vec{B}_t = ik\vec{\nabla}_t B_z,$$

so $\vec{E}_t = -\omega\hat{z}\times\vec{B}_t/k$. Premultiplying by $\hat{z}\times$, we have $H_t = k\hat{z}\times E_t/\mu\omega$. In both cases we have

$$\vec{H}_t = \frac{1}{Z}\hat{z}\times\vec{E}_t, \qquad Z = \begin{cases} \frac{k}{\epsilon\omega} = \frac{k}{k_0}\sqrt{\frac{\mu}{\epsilon}} & \text{TM} \\ \frac{\mu\omega}{k} = \frac{k_0}{k}\sqrt{\frac{\mu}{\epsilon}} & \text{TE} \end{cases}$$

Now each component of $\vec{E}$ and $\vec{B}$ is of the form $\Psi(x,y)e^{ikz-i\omega t}$ where each Ψ satisfies

$$\left(\nabla_t^2 + \gamma^2\right)\Psi = 0 \qquad \gamma^2 = \mu\epsilon\omega^2 - k^2$$

For the TM and TE modes they are determined by a single scalar ψ, and for $t = z = 0$ are given by

$$\begin{array}{llll} \text{TM:} & E_z = \psi, & \vec{E}_t = ik\gamma^{-2}\vec{\nabla}_t\psi & \psi|_\Gamma = 0 \\ \text{TE:} & H_z = \psi, & \vec{H}_t = ik\gamma^{-2}\vec{\nabla}_t\psi & \hat{n}\cdot\vec{\nabla}_t\psi|_\Gamma = 0, \end{array}$$

with $(\nabla_t^2 + \gamma^2)\psi = 0$. Note that with these conditions,

$$0 = \int_A \psi^*\left(\nabla_t^2 + \gamma^2\right)\psi = \int_A \vec{\nabla}_t\cdot(\psi^*\vec{\nabla}_t\psi) - \int_A (\vec{\nabla}_t\psi)^*\cdot\vec{\nabla}_t\psi + \gamma^2\int_A |\psi|^2,$$

where A is the cross section. The first integral is a divergence, so is $\oint_{\partial A}\psi^*\vec{n}\cdot\vec{\nabla}_t\psi$, which vanishes from either boundary condition, the second integral is strictly positive unless ψ is a constant[1], and the coefficient of γ^2 is positive, so γ^2 is positive. There will be solutions of the two-dimensional Helmholtz equation for discrete positive values γ_λ^2. For each frequency ω, there can be waves with wave numbers

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2,$$

so only waves with $\omega > \omega_\lambda := \gamma_\lambda/\sqrt{\mu\epsilon}$ can propagate. With $k_\lambda^2 < 0$ we can have cutoff modes (or evanescent modes) which do note propagate but decay with z. Note $k_\lambda < \sqrt{\mu\epsilon}\,\omega$, the value the wavelength would have in an infinite medium, so the wavelength in the waveguide is longer than in $\mathbb{R}^3$. The phase velocity $v_p = \omega/k_\lambda > 1/\sqrt{\mu\epsilon}$, greater than in $\mathbb{R}^3$.

[1] In which case we must have a TE mode, but then E_t and B_t are both zero, $\vec{E} = 0$, and thus $B_z =$ constant.

1.5 An Example

We see that finding the dispersion of a cylindrical waveguide involves solving the two dimensional Helmholtz equation with boundary conditions specified on Γ, the cross section's intersection with the surface.

$$\left(\nabla_t^2 + \gamma^2\right)\psi = 0 \quad \text{with} \quad \psi|_\Gamma = 0 \text{ (TM)} \quad \text{or } \hat{n}\cdot\vec{\nabla}\psi|_\Gamma = 0 \text{ (TE)}.$$

There are a number of coordinate systems for which the Laplacian operator can be separated, and if the boundary shapes are suitable, it is straightforward to find solutions. Of course the simplest is a rectangular wave guide, for which we can use cartesian coordinates. This is worked out in Jackson, section 8.4, and you should definitely work through it (rectangular waveguides have appeared on the qualifier!), but it is quite clear and it would add nothing for me to repeat the solution, so instead, lets consider a circular cylindrical waveguide of radius r.

Naturally we should use cylindrical coordinates, or for the cross section simply polar coordinates ρ, ϕ. The Laplace operator in polar coordinates is

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}.$$

If we make an ansatz that the solution $\psi(\rho,\phi) = R(\rho)\Phi(\phi)$, we have

$$\left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2 R(\rho)\right)\Phi(\phi) + \frac{1}{\rho^2}R(\rho)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

Dividing by $R(\rho)\Phi(\phi)$ and multiplying by ρ^2 gives

$$\begin{aligned} \frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right)& \\ +\frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} &= 0. \end{aligned}$$

The first line depends on ρ but not on ϕ, which the second depends on ϕ but not on ρ, so they must be equal and opposite constants,

$$\begin{aligned} \frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) &= C \\ \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} &= -C. \end{aligned}$$

The second equation,

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2} + C\Phi(\phi) = 0$$

has the solution $\Phi(\phi) = e^{\pm i\sqrt{C}\phi}$. As we need a solution periodic in ϕ, that is $\Phi(\phi+2\pi) = \Phi(\phi)$, we see that $\sqrt{C}$ must be an integer, m. Then the first equation is

$$\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \gamma^2\rho^2 - m^2\right)R(\rho) = 0,$$

which is the Bessel equation, with solutions regular at the origin given by $J_m(\gamma\rho)$.

It is straightforward to satisfy the boundary condition by demanding that γr is a zero of J_m (for TM waves) or of $dJ_m(x)/dx$ (for TE waves). These can be looked up in many books[2]. We have $x_{mn}^{\rm TM}$ the n'th zero of J_m and $x_{mn}^{\rm TE}$ the n'th zero of J'_m. In terms of that numerical value, x_{mn}, we have $\gamma = x_{mn}/r$, and the tube can only support electromagnetic waves with a frequency greater than the cutoff frequency $\omega_{mn} = x_{mn}/r\sqrt{\mu\epsilon}$. The smallest of these roots is that J'_1, with $x_{11}^{\rm TE} = 1.8412$, and the next is that of J_0, with $x_{01}^{\rm TM} = 2.4048$. If the waveguide is 5 cm in diameter, and filled with air $\sim$ vacuum, this gives a cutoff on TE modes of $f = \frac{\omega}{2\pi} = 3.5$ GHz and 4.6 GHz for the lowest TM mode.

[2]For example, Arfken III p. 581, or Jackson p. 114 and 370, or, for far more, Abramowitz and Stegun, p. 409 and 411.