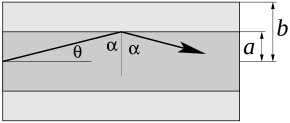
Fiber Optics

Waves can be guided not only by conductors, but by dielectrics. Fiber optics cable of silica has $n(r)$ varying with radius.

Simplest: core radius a with $n = n_1$, surrounded (radius b) with $n = n_0 < n_1$.
Total internal reflection if



$$\alpha > \alpha_c = \sin^{-1}(n_0/n_1)$$

Equivalently $\theta < \theta_{\max} = \cos^{-1}(n_0/n_1)$.
This is geometrical optics. Needs $\lambda \ll a$.

Two kinds of fibers:

- multimode, $a \gg \lambda$, treat with geometrical optics. Typically $a \approx 25\,\mu$m, $b \approx 75\,\mu$m, $\lambda \sim 0.85\,\mu$m.
- single mode, $a \sim \lambda$, treat as wave guide. Typically $a \approx 2\,\mu$m.

Multimode fibers

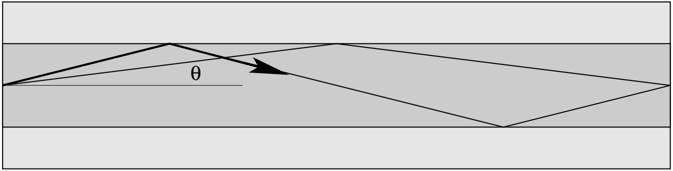
Define $\Delta = \dfrac{n_1^2 - n_0^2}{2n_1^2} \approx 1 - \dfrac{n_0}{n_1}$, typically about 0.01.
$\cos\theta_{\max} \approx 1 - \frac{1}{2}\theta_{\max}^2 = 1 - \Delta$, so $\theta_{\max} \approx \sqrt{2\Delta}$.
How many modes can propagate?
Uncertainty principle: only one mode can fit per unit "volume" in phase space, $N = \int \left(\dfrac{dpdq}{2\pi\hbar}\right)^D$ for each mode in D dimensions. Here $D = 2$, the cross section has coordinate integral $\int d^2q = \pi a^2$. As
$|\vec{k}_\perp| \leq k_z \tan\theta_{\max} = k_z\sqrt{2\Delta}$,
$\int d^2p = \hbar^2 \int d^2k = 2\pi\hbar^2 k_z^2\Delta$. There are two polarizations, so

$$N = 2\frac{1}{(2\pi)^2}\left(\pi a^2\right)\left(2\pi k_z^2\Delta\right) = \frac{1}{2}V^2,$$

where$V := ka\sqrt{2\Delta}$ is called the *fiber parameter*.

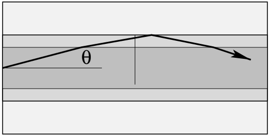
Problem with simple fiber

At each angle $\theta < \theta_{\mathrm{max}}$, light travels indefinitely down the fiber. But to go a large distance L down the fiber,



it travels a different distance $L \sec\theta$, so light from different θ's arrive with different phases, and interfere!

Fix: make several transitions to lower n. In fact, for homework (Jackson 8.14) you will find a "perfect" fix, using n varying continuously with radius.



$n(x)$ varying with radius x

Consider dielectric with $\epsilon(\vec{x})$ varying smoothly, $\mu = \mu_0$ as silica is not magnetic. Assume single frequency ω.
Maxwell gives

$$\begin{aligned} \vec{\nabla} \cdot \epsilon \vec{E} &= 0 = \left(\vec{\nabla} \epsilon\right) \cdot \vec{E} + \epsilon \vec{\nabla} \cdot \vec{E} \\ \vec{\nabla} \times \vec{E} &= -\mu_0 \frac{\partial \vec{H}}{\partial t} = i \mu_0 \omega \vec{H} \\ \vec{\nabla} \times \vec{H} &= \frac{\partial \epsilon \vec{E}}{\partial t} = -i \omega \epsilon \vec{E} \\ \vec{\nabla} \cdot \vec{H} &= 0. \end{aligned}$$

So

$$\begin{aligned} \vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right) &= -\nabla^2 \vec{E} + \vec{\nabla}\left(\vec{\nabla} \cdot \vec{E}\right) = i \mu_0 \omega \vec{\nabla} \times \vec{H} \\ &= \mu_0 \omega^2 \epsilon \vec{E} \\ &= -\nabla^2 \vec{E} - \vec{\nabla}\left(\frac{1}{\epsilon}\left(\vec{\nabla} \epsilon\right) \cdot \vec{E}\right) \end{aligned}$$

The same for H:

$$\begin{aligned}\vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right) &= -\nabla^2\vec{H}+\vec{\nabla}\left(\vec{\nabla}\cdot\vec{H}\right) = -i\omega\vec{\nabla}\times\left(\epsilon\vec{E}\right)\\ -\nabla^2\vec{H} &= -i\omega\left(\vec{\nabla}\epsilon\right)\times\vec{E} - i\omega\epsilon\vec{\nabla}\times\vec{E}\\ &= -i\omega\left(\vec{\nabla}\epsilon\right)\times\vec{E}+\mu_0\omega^2\epsilon\vec{H}.\end{aligned}$$

Thus
$$\begin{aligned}\nabla^2\vec{E}+\mu_0\omega^2\epsilon\vec{E}+\vec{\nabla}\left(\frac{1}{\epsilon}\left(\vec{\nabla}\epsilon\right)\cdot\vec{E}\right) &= 0\\ \nabla^2\vec{H}+\mu_0\omega^2\epsilon\vec{H}-i\omega\left(\vec{\nabla}\epsilon\right)\times\vec{E} &= 0\end{aligned}$$

Assume ϵ varies slowly compared to λ,

$$\nabla\epsilon \ll \frac{\epsilon}{\lambda} = \frac{\epsilon\omega}{c}.$$

Other terms are ω^2/c^2 times E or H, but $\nabla\epsilon$ terms are $\nabla\epsilon/\epsilon\lambda$ times E, $\ll \lambda^2 = \omega^2/c^2$, so they can be ignored. Both $\vec{E}$ and $\vec{H}$ satisfy

$$\left(\nabla^2+\frac{\omega^2}{c^2}n^2(\vec{r})\right)\psi(\vec{r}) = 0.$$

Eikonal

ψ oscillates rapidly (on scale $\sim \lambda$). Take this away by defining the eikonal $S(\vec{r})$, with

$$\psi(\vec{r}) = e^{i\omega S(\vec{r})/c}$$

$$\text{so} \quad \nabla^2\psi = \vec{\nabla}\cdot\left(\frac{i\omega}{c}\vec{\nabla}S e^{i\omega S(\vec{r})/c}\right)$$
$$= \left[\frac{i\omega}{c}\nabla^2 S - i\left(\frac{\omega}{c}\right)^2\left(\vec{\nabla}S\right)^2\right]e^{i\omega S(\vec{r})/c}$$
$$= -(\omega^2 n^2/c^2)e^{i\omega S/c}$$

and $n^2(\vec{r}) - \vec{\nabla}S\cdot\vec{\nabla}S = -i\dfrac{c}{\omega}\nabla^2 S$. Now $c/\omega \sim \lambda$ while ∇S varies with $n(\vec{r})$, much more slowly, so we can set the r.h.s to zero, for the
Eikonal approximation: $\quad \vec{\nabla}S\cdot\vec{\nabla}S = n^2(\vec{r})$.

The *Eikonal approximation*: $\vec{\nabla} S \cdot \vec{\nabla} S = n^2(\vec{r})$ doesn't give the direction S changes, but does give the rate.

Define $\hat{k}(\vec{r})$ so $\vec{\nabla} S = n(\vec{r})\hat{k}(\vec{r})$. Near a point r_0,

$$\begin{aligned}\psi(\vec{r}) &\approx e^{i\omega\left(S(\vec{r}_0) + (\vec{r}-\vec{r}_0)\cdot\vec{\nabla} S\right)/c} \\ &= e^{i\omega S(\vec{r}_0)/c}\, e^{i\omega\hat{k}\cdot(\vec{r}-\vec{r}_0)n(\vec{r})/c},\end{aligned}$$

so it is locally a plane wave with $|\vec{k}| = \omega n(\vec{r})/c$.

Consider an integral curve, that is, a ray following $\vec{\nabla} S$, and let s be the distance along that curve. Then $d\vec{r}/ds = \hat{k}$, $n(\vec{r})d\vec{r}/ds = \vec{\nabla} S$, so

$$\frac{d}{ds}\left(n(\vec{r})\frac{d\vec{r}}{ds}\right) = \frac{d}{ds}\vec{\nabla} S = \vec{\nabla}\left.\frac{dS}{ds}\right|_{\Gamma} = \vec{\nabla} n(\vec{r}). \qquad (1)$$

Meridional rays pass through axis ($m = 0$ as waves)
skew rays do not, travel helically ($m \neq 0$ modes).

We will treat only meridional, effectively in xz plane, with x a radial direction and z along the fiber. We assume $n(\vec{r}) = n(x)$ independent of z.
Take x and z components of

$$(1): \quad \frac{d}{ds}\left(n(\vec{r})\frac{d\vec{r}}{ds}\right) = \vec{\nabla}n(\vec{r})$$

$$\frac{d}{ds}\left(n(x)\sin\theta\right) = \frac{dn(x)}{dx}, \qquad \frac{d}{ds}\left(n(x)\cos\theta\right) = \frac{dn(\vec{r})}{dz} = 0.$$

So $n(x)\cos\theta =$ constant. With $\theta(0) < \theta_{\text{max}}$ ray reaches a maximum radius x_{max} with $\bar{n} := n(0)\cos\theta(0) = n(x_{\text{max}})$.

$$\frac{dz}{ds} = \cos\theta = \frac{\bar{n}}{n(x)}, \qquad \frac{d}{ds} = \frac{\bar{n}}{n(x)}\frac{d}{dz},$$

so the x component of (1) gives

$$\frac{dn}{dx} = \frac{\bar{n}}{n(x)}\frac{d}{dz}\left(n(x)\frac{\bar{n}}{n(x)}\frac{dx}{dz}\right) = \frac{\bar{n}}{n(x)}\frac{d}{dz}\left(\bar{n}\frac{dx}{dz}\right),$$

$$\text{so} \qquad \bar{n}^2\frac{d^2x}{dz^2} = n(x)\frac{dn(x)}{dx} = \frac{1}{2}\frac{d}{dx}n^2(x).$$

$$\bar{n}^2 \frac{d^2 x}{dz^2} = \frac{1}{2} \frac{d}{dx} n^2(x)$$

Looks like $ma = -dV/dx$ with potential $-\frac{1}{2}n^2(x)$ and time z, so as for Newton, multiply by "velocity" dx/dz, to get

$$\frac{1}{2}\bar{n}^2 \frac{d}{dz}\left(\frac{dx}{dz}\right)^2 = \frac{1}{2}\frac{d}{dz}n^2(x) \Longrightarrow \bar{n}^2 \underbrace{\left(\frac{dx}{dz}\right)^2}_{=0 \text{ at } x_{\max}} = n^2(x) - \bar{n}^2.$$

The distance travelled along z in getting from the axis to x is

$$z(x) = \int_0^x \frac{dz}{dx} dx = \bar{n} \int_0^x \frac{dx}{\sqrt{n^2(x) - \bar{n}^2}},$$

and the distance from one axis crossing to the next is

$$Z = 2\bar{n} \int_0^{x\max} \frac{dx}{\sqrt{n^2(x) - \bar{n}^2}}.$$

The optical distance $\int n(x)ds$ between axis crossings is

$$\begin{aligned} L_{\rm opt} &= 2\int_0^{x{\rm max}} n(x)\frac{ds}{dz}\frac{dz}{dx}\,dx \\ &= 2\int_0^{x{\rm max}} n(x)\frac{n(x)}{\bar n}\frac{\bar n}{\sqrt{n^2(x)-\bar n^2}}\,dx \\ &= 2\int_0^{x{\rm max}} \frac{n^2(x)}{\sqrt{n^2(x)-\bar n^2}}\,dx. \end{aligned}$$

Over a long distance L, many axis crossings (L/Z), total phase change is proportional to $L\frac{L_{\rm opt}}{Z}$. It is ideal if $\frac{L_{\rm opt}}{Z}$ is independent of $\bar n$, for otherwise different rays will destructively interfere.
You will find the ideal in problem 8.14.
Signals will also degrade with distance if there is dispersion over the bandwidth of the signal. There is also some absorption in real dielectrics. These two issues for silica favor using $\lambda \sim 1.4\mu$m.