Generalized Coordinates

Cartesian coordinates $r^i, i = 1, 2, \dots D$ for Euclidean space.
Distance by Pythagoras: $(\delta s)^2 = \sum_i (\delta r^i)^2$.

Unit vectors $\hat{e}_i$, displacement $\Delta\vec{r} = \sum_i \Delta r^i \hat{e}_i$
Fields are functions of position, or of $\vec{r}$ or of $\{r^i\}$.
Scalar fields $\Phi(\vec{r})$, Vector fields $\vec{V}(\vec{r})$

$$\begin{aligned}
\vec{\nabla}\Phi &= \sum_i \frac{\partial \Phi}{\partial r^i}\hat{e}_i, \\
\vec{\nabla}\cdot\vec{V} &= \sum_i \frac{\partial V_i}{\partial r^i}, \\
\nabla^2\Phi = \vec{\nabla}\cdot\vec{\nabla}\Phi &= \sum_i \frac{\partial^2 \Phi}{\partial {r^i}^2}, \\
\vec{\nabla}\times\vec{V} &= \sum_{ijk} \epsilon_{ijk}\frac{\partial V_k}{\partial r^j}\hat{e}_i, \qquad \text{3D only}
\end{aligned}$$

Other smooth coordinatization $q^i, i = 1, \dots D$
$q^i(\vec{r})$ and $\vec{r}(\{q^i\})$ are well defined (in some domain) mostly 1—1, so Jacobian $\det(\partial q^i/\partial r^j) \neq 0$.

Distance between $P \Leftarrow\!\!\Rightarrow \{q^i\}$ and $P' \Leftarrow\!\!\Rightarrow \{q^i + \delta q^i\}$ is given by

$$\begin{aligned}
(\delta s)^2 &= \sum_k (\delta r^k)^2 = \sum_k \left(\sum_i \frac{\partial r^k}{\partial q^i}\delta q^i\right)\left(\sum_j \frac{\partial r^k}{\partial q^j}\delta q^j\right) \\
&= \sum_{ij} g_{ij}\delta q^i \delta q^j,
\end{aligned}$$

where

$$g_{ij} = \sum_k \frac{\partial r^k}{\partial q^i}\frac{\partial r^k}{\partial q^j}.$$

g_{ij} is a real symmetric matrix called the *metric tensor*.
In general a nontrivial function of the position, $g_{ij}(q)$.
To repeat:

$$(\delta s)^2 = \sum_{ij} g_{ij}\delta q^i \delta q^j.$$

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Functions (fields)

A scalar field $f(P) \Leftarrow\!\!\Rightarrow f(\vec{r})$ can also be specified by a function of the q's, $\tilde{f}(q) = f(\vec{r}(q))$.
What about vector fields? $\vec{V}(\vec{r})$ has a meaning independent of the coordinates used to describe it, but components depend on the basis vectors. Should have basis vectors $\tilde{e}_i$ aligned with the direction of q^i. How to define?
Consider

$$\tilde{e}_1 = \lim_{\delta q^1 \to 0} \frac{\vec{r}(q^1 + \delta q^1, q^2, q^3) - \vec{r}(q^1, q^2, q^3)}{\delta s} = \sum_k \frac{\partial r^k}{\partial q^1}\frac{\hat{e}_k}{\sqrt{g_{11}}}$$

and the similarly defined $\tilde{e}_2$ and $\tilde{e}_3$. In general not good orthonormal bases, because

$$\tilde{e}_1 \cdot \tilde{e}_2 = \sum_k \frac{\partial r^k}{\partial q^1}\frac{\partial r^k}{\partial q^2}/\sqrt{g_{11}g_{22}} = g_{12}/\sqrt{g_{11}g_{22}},$$

which need not be zero.

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Another problem: $\tilde{e}_k$ is not normal to the surface $q^k =$ constant.

What to do? Two approaches:

- Give up on orthonormal basis vectors. Define differential forms, such as $d\Phi = \sum_k \frac{\partial \Phi}{\partial x^k}dx^k$. The coefficients of dx^k transform as covariant vectors. Contravariant vectors may be considered coefficients of directional derivatives $\partial/\partial q^k$. This is the favored approach for working in curved spaces, differential geometry and general relativity.
- Restrict ourselves to *orthogonal coordinate systems* — surface $q^i =$ constant intersects $q^j =$ constant at right angles. Then $\vec{\nabla}q^i \cdot \vec{\nabla}q^j = 0$.

In general define $\qquad g^{ij} := \vec{\nabla}q^i \cdot \vec{\nabla}q^j = \sum_k \frac{\partial q^i}{\partial r^k}\frac{\partial q^j}{\partial r^k}$.

Note that g^{ij} is not the same as g_{ij}.

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In fact

$$\sum_\ell g^{i\ell}g_{\ell j} = \sum_\ell \sum_k \frac{\partial q^i}{\partial r^k}\frac{\partial q^\ell}{\partial r^k}\sum_m \frac{\partial r^m}{\partial q^\ell}\frac{\partial r^m}{\partial q^j} = \sum_{km}\frac{\partial q^i}{\partial r^k}\delta_{km}\frac{\partial r^m}{\partial q^j}$$

$= \delta_{ij}$, so $g^{\cdot\cdot}$ is the inverse matrix to $g_{\cdot\cdot}$.

If $\vec{\nabla}q^i \cdot \vec{\nabla}q^j = 0$ for $i \neq j$, $g^{ij} = 0$ for $i \neq j$, $g^{\cdot\cdot}$ is diagonal, so $g_{\cdot\cdot}$ is also diagonal. And as $(\delta s)^2 > 0$ for any non-zero $\delta\vec{r}$, $g_{\cdot\cdot}$ is positive definite, so for an orthogonal coordinate system the diagonal elements are positive, $g_{ij} = h_i^2\delta_{ij}$, and $g^{ij} = h_i^{-2}\delta_{ij}$. Then the unit vectors are

$$\tilde{e}_i = h_i^{-1}\sum_k \hat{e}_k \frac{\partial r^k}{\partial q^i} =: \sum_k B_{ki}\hat{e}_k \tag{1}$$

with the inverse relation

$$\hat{e}_k = \sum_i h_i \tilde{e}_i \frac{\partial q^i}{\partial r^k} =: \sum_i A_{ki}\tilde{e}_i = \sum_i A_{ki}\sum_\ell B_{\ell i}\hat{e}_\ell. \tag{2}$$

As the $\hat{e}_k$ are independent, this implies $\sum_i A_{ki}B_{\ell i} = \delta_{k\ell}$ or $AB^T = 1\!\!1$.

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The set $\{\hat{e}_k\}$ and the set $\{\tilde{e}_i\}$ are each orthonormal, and $\hat{e}_k = \sum_i A_{ki}\tilde{e}_i$, so A is orthogonal, and $\therefore B = A$.
Can also check as

$$\sum_k A_{ki}A_{kj} = h_i h_j \sum_k (\partial q^i/\partial r^k)(\partial q^j/\partial r^k) = h_i h_j g^{ij} = \delta_{ij}.$$

Thus A can be written two ways,

$$A_{ki} = h_i \frac{\partial q^i}{\partial r^k} = h_i^{-1}\frac{\partial r^k}{\partial q^i}.$$

Note that A_{jk} is a function of position, not a constant. In Euclidean space we say that $\hat{e}_x$ is the same vector regardless of which point $\vec{r}$ the vector is at[1]. But then $\tilde{e}_i = \sum_j A_{ji}(P)\hat{e}_j$ is not the same vector at different points P.

[1] That is, Euclidean space comes with a prescribed *parallel transport*, telling how to move a vector without changing it.

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Vector Fields

So for a vector field

$$\vec{V}(P)=\sum_j \tilde{V}_j(q)\tilde{e}_j=\sum_i V_i(\vec{r})\hat{e}_i=\sum_{ik} V_i(\vec{r})A_{ik}\tilde{e}_k,$$

so $\tilde{V}_k(q)=\sum_i V_i(\vec{r}(q))A_{ik}(q)$.
Also $V_k(\vec{r})=\sum_i A_{ki}(\vec{r})\tilde{V}_i(q(\vec{r}))$.

Let's summarize some of our previous relations:

$$\hat{e}_k=\sum_i A_{ki}\tilde{e}_i, \qquad \tilde{e}_i=\sum_j A_{ji}(P)\hat{e}_j$$

$$A_{ki}=h_i\frac{\partial q^i}{\partial r^k}=h_i^{-1}\frac{\partial r^k}{\partial q^i}.$$

$$g_{ij}=h_i^2\delta_{ij}, \qquad g^{ij}=h_i^{-2}\delta_{ij}$$

Gradient of a scalar field

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$$\begin{aligned}\vec{\nabla} f &= \sum_k \frac{\partial f}{\partial r^k}\hat{e}_k=\sum_{k\ell m}\left(\frac{\partial\tilde{f}}{\partial q^\ell}\frac{\partial q^\ell}{\partial r^k}\right)(A_{km}\tilde{e}_m)\\ &=\sum_{k\ell m}\left(\frac{\partial \tilde{f}}{\partial q^\ell}\frac{\partial q^\ell}{\partial r^k}\right)\left(h_m\tilde{e}_m\frac{\partial q^m}{\partial r^k}\right)=\sum_{\ell m}\frac{\partial\tilde{f}}{\partial q^\ell}h_m\tilde{e}_m g^{m\ell}\\ &=\sum_{\ell m}\frac{\partial \tilde{f}}{\partial q^\ell}h_m\tilde{e}_m h_m^{-2}\delta_{m\ell}=\sum_m h_m^{-1}\frac{\partial\tilde{f}}{\partial q^m}\tilde{e}_m,\end{aligned}$$

or

$$\vec{\nabla} f = \sum_m h_m^{-1}\frac{\partial \tilde{f}}{\partial q^m}\tilde{e}_m. \tag{3}$$

Both $f(\vec{r})$ and $\tilde{f}(q)$ represent the same function $f(P)$ on space, so we often carelessly leave out the twiddle.

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Velocity of a particle

$$\begin{aligned}\vec{v} &= \sum_k \frac{dr^k}{dt}\hat{e}_k=\sum_k\left(\sum_i\frac{\partial r^k}{\partial q^i}\frac{dq^i}{dt}\right)\left(\sum_j h_j\frac{\partial q^j}{\partial r^k}\tilde{e}_j\right)\\ &=\sum_{ij}\frac{dq^i}{dt}h_j\delta_{ij}\tilde{e}_j=\sum_j h_j\frac{dq^j}{dt}\tilde{e}_j.\end{aligned}$$

Note it is $h_j dq^j$ which has the right dimensions for an infinitesimal *length*, while dq^j by itself might not.

Example, spherical coordinates.

Surfaces of constant $\begin{cases} r & \text{spherical shells}\\ \theta & \text{cone, vertex at 0}\\ \phi & \text{plane containing } z\end{cases}$

with the shells centered at the origin. These intersect at right angles, so they are orthogonal coords.

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By looking at distances from varying one coordinate, comparing to $(ds)^2=h_r^2(dr)^2+h_\theta^2(d\theta)^2+h_\phi^2(d\phi)^2$, we see that

$$h_r=1, \quad h_\theta=r, \quad h_\phi=r\sin\theta.$$

Thus

$$\vec{v}=\dot{r}\tilde{e}_r+r\dot{\theta}\tilde{e}_\theta+r\sin\theta\dot{\phi}\tilde{e}_\phi$$

and

$$v^2=\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2.$$

Derivatives of Vectors

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Euclidean parallel transport: $\dfrac{\partial \hat{e}_i}{\partial r^j}=0$

$$\begin{aligned}h_j^{-1}\frac{\partial}{\partial q^j}\tilde{e}_i &= h_j^{-1}\sum_{k\ell}\frac{\partial r^k}{\partial q^j}\frac{\partial}{\partial r^k}\left(h_i^{-1}\hat{e}_\ell\frac{\partial r^\ell}{\partial q^i}\right)\\ &= h_j^{-1}\sum_{k\ell}\frac{\partial r^k}{\partial q^j}\frac{\partial A_{\ell i}}{\partial r^k}\hat{e}_\ell\\ &= \sum_{k\ell}A_{kj}\frac{\partial A_{\ell i}}{\partial r^k}\hat{e}_\ell.\end{aligned}$$

Differential Forms

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A general 1-form over a space coordinatized by q^i is

$$\omega=\sum_i A_i(P)dq^i,$$

and is associated with a vector. But if we are using orthogonal curvilinear coordinates, it is more natural to express the coefficients as multiplying the "normalized" 1-forms $\omega_i:=h_i dq^i$, with $\omega=\sum_i A_i dq^i=\tilde{V}_i\omega_i$.
Then ω is associated with the vector $\vec{V}=\sum_i\tilde{V}_i\tilde{e}_i$.
Note that if

$$\omega=df=\sigma_i\frac{\partial f}{\partial q^i}dq^i=\sum_i h_i^{-1}\frac{\partial f}{\partial q^i}\omega_i,$$

the associated vector is $\vec{V}=\sum_i h_i^{-1}\dfrac{\partial f}{\partial q^i}\tilde{e}_i=\vec{\nabla} f$.

2-forms

General 2-form: $\omega^{(2)} = \frac{1}{2}\sum_{ij} B_{ij}\omega_i \wedge \omega_j$, with $B_{ij} = -B_{ji}$.

In three dimensions, this can be associated with a vector $\vec{B} = \sum_i \tilde{B}_i \tilde{e}_i$ with $\tilde{B}_i = \frac{1}{2}\sum_{jk}\epsilon_{ijk}B_{jk}$, $\quad B_{jk} = \sum_i \epsilon_{ijk}\tilde{B}_i$.

If $\vec{V} \rightleftharpoons \omega^{(1)}$ and $\omega^{(2)} = d\omega^{(1)}$, then

$$\begin{aligned}
\omega^{(2)} &= \frac{1}{2}\sum_{ij} B_{ij}\omega_i \wedge \omega_j = d\left(\sum_i \tilde{V}_i h_i dq^i\right) \\
&= \sum_{ij} \frac{\partial(\tilde{V}_i h_i)}{\partial q^j} dq^j \wedge dq^i \\
&= \sum_{ij} h_i^{-1} h_j^{-1} \frac{\partial(\tilde{V}_i h_i)}{\partial q^j} \omega_j \wedge \omega_i,
\end{aligned}$$

Thus $\frac{1}{2}B_{ij} = \frac{1}{2}h_i^{-1}h_j^{-1}\left(\frac{\partial(\tilde{V}_j h_j)}{\partial q^i} - \frac{\partial(\tilde{V}_i h_i)}{\partial q^j}\right)$.

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The associated vector has coefficients

$$\begin{aligned}
\tilde{B}_k &= \frac{1}{2}\sum_{ij}\epsilon_{ijk}\frac{1}{h_i h_j}\left(\frac{\partial}{\partial q^i}\tilde{V}_j h_j - \frac{\partial}{\partial q^j}\tilde{V}_i h_i\right) \\
&= \sum_{ij}\epsilon_{ijk}\frac{1}{h_i h_j}\frac{\partial}{\partial q^i}\left(\tilde{V}_j h_j\right).
\end{aligned}$$

For cartesian coordinates $h_i = 1$, and we recognize this as the curl of $\vec{V}$, so $d\omega^{(1)} \rightleftharpoons \vec{\nabla}\times\vec{V}$, which is a coordinate-independent statement. Thus we have for any orthogonal curvilinear coordinates

$$\vec{\nabla}\times\sum_i \tilde{V}_i\tilde{e}_i = \sum_{ijk}\epsilon_{ijk}\frac{1}{h_i h_j}\frac{\partial}{\partial q^i}\left(h_j\tilde{V}_j\right)\tilde{e}_k. \tag{4}$$

$\vec{\nabla}\cdot\vec{B}$ and 3-forms

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Finally, let's consider a vector $\vec{B} = \sum_i \tilde{B}_i\tilde{e}_i$ and its associated 2-form

$$\omega^{(2)} = \frac{1}{2}\sum_i \epsilon_{ijk}\tilde{B}_i\omega_j\wedge\omega_k = \frac{1}{2}\sum_i \epsilon_{ijk}\tilde{B}_i h_j h_k dq^j\wedge dq^k.$$

The exterior derivative is a three-form

$$\begin{aligned}
d\omega^{(2)} &= \frac{1}{2}\sum_{ijk\ell}\epsilon_{ijk}\frac{\partial\tilde{B}_i h_j h_k}{\partial q_\ell}dq^\ell\wedge dq^j\wedge dq^k \\
&= \frac{1}{2}\sum_{ijk\ell}\epsilon_{ijk\ell}\epsilon_{\ell jk}\frac{\partial\tilde{B}_i h_j h_k}{\partial q_\ell}dq^1\wedge dq^2\wedge dq^3 \\
&= \frac{1}{h_1h_2h_3}\sum_i\frac{\partial}{\partial q_i}\left(\frac{h_1h_2h_3}{h_i}\tilde{B}_i\right)\omega_1\wedge\omega_2\wedge\omega_3.
\end{aligned}$$

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A 3-form in three dimensions is associated with a scalar function $f(P)$ by

$$\begin{aligned}
\omega^{(3)} &= f\,dr^1\wedge dr^2\wedge dr^3 = \frac{1}{6}f\sum_{abc}\epsilon_{abc}dr^a\wedge dr^b\wedge dr^c \\
&= \frac{1}{6}f\sum_{abcijk}\epsilon_{abc}\frac{\partial r^a}{\partial q^i}\frac{\partial r^b}{\partial q^j}\frac{\partial r^c}{\partial q^k}dq^i\wedge dq^j\wedge dq^k \\
&= \frac{1}{6}f\sum_{abcijk}\epsilon_{abc}\left(h_i^{-1}\frac{\partial r^a}{\partial q^i}\right)\left(h_j^{-1}\frac{\partial r^b}{\partial q^j}\right)\left(h_k^{-1}\frac{\partial r^c}{\partial q^k}\right) \\
&\qquad \omega_i\wedge\omega_j\wedge\omega_k \\
&= \frac{1}{6}f\det A\sum_{ijk}\epsilon_{ijk}\omega_i\wedge\omega_j\wedge\omega_k \\
&= f\det A\;\omega_1\wedge\omega_2\wedge\omega_3 \\
&= f\;\omega_1\wedge\omega_2\wedge\omega_3,
\end{aligned}$$

where $\det A = 1$ because A is orthogonal, but also we assume the $\tilde{e}_i$ form a right handed coordiate system.

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So we see that if $\omega^{(2)} \rightleftharpoons \vec{B}$, $d\omega^{(2)} \rightleftharpoons f$, with

$$f = \frac{1}{h_1h_2h_3}\sum_i\frac{\partial}{\partial q_i}\left(\frac{h_1h_2h_3}{h_i}\tilde{B}_i\right).$$

In cartesian coordinates this just reduces to $\vec{\nabla}\cdot\vec{B}$, but this association is coordinate-independent, so we see that in a general curvilinear coordinate system,

$$\vec{\nabla}\cdot\vec{B} = \frac{1}{h_1h_2h_3}\sum_i\frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i}\tilde{B}_i\right).$$

Finally, let's examine the Laplacian on a scalar:

$$\begin{aligned}
f &= \nabla^2\Phi = \vec{\nabla}\cdot\vec{\nabla}\Phi = \frac{1}{h_1h_2h_3}\sum_i\frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i}h_i^{-1}\frac{\partial\Phi}{\partial q^i}\right) \\
&= \frac{1}{h_1h_2h_3}\sum_i\frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\Phi}{\partial q^i}\right)
\end{aligned}$$

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$$\vec{v} = \sum_j h_j\frac{dq^j}{dt}\tilde{e}_j, \qquad \vec{\nabla}f = \sum_m h_m^{-1}\frac{\partial\tilde{f}}{\partial q^m}\tilde{e}_m,$$

$$\begin{aligned}
\vec{\nabla}\times\left(\sum_i\tilde{V}_i\tilde{e}_i\right) &= \sum_{ijk}\epsilon_{ijk}\frac{1}{h_ih_j}\frac{\partial}{\partial q^i}\left(h_j\tilde{V}_j\right)\tilde{e}_k, \\
\vec{\nabla}\cdot\vec{B} &= \frac{1}{h_1h_2h_3}\sum_i\frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i}\tilde{B}_i\right), \\
\nabla^2\Phi &= \frac{1}{h_1h_2h_3}\sum_i\frac{\partial}{\partial q^i}\left(\frac{h_1h_2h_3}{h_i^2}\frac{\partial\Phi}{\partial q^i}\right)
\end{aligned}$$

For the record, even for generalized coordinates that are not orthogonal, we can write

$$\nabla^2 = \frac{1}{\sqrt{g}}\sum_{ij}\frac{\partial}{\partial q^i}g^{ij}\sqrt{g}\frac{\partial}{\partial q^j},$$

where $g := \det g_{\cdot\cdot}$.

Cylindrical Polar Coordinates

Although we have developed this to deal with esoteric orthogonal coordinate systems, such as those for your homework, let us here work out the familiar cylindrical polar and spherical coordinate systems.

Cylindrical Polar: r, ϕ, z,

$$(\delta s)^2 = (\delta r)^2 + r^2(\delta\phi)^2 + (\delta z)^2,$$

so $h_r = h_z = 1, h_\phi = r$. Then

$$\vec\nabla f = \frac{\partial f}{\partial r}\tilde e_r + \frac{1}{r}\frac{\partial f}{\partial \phi}\tilde e_\phi + \frac{\partial f}{\partial z}\tilde e_z,$$

Polar, continued

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$$\begin{aligned}
\vec\nabla\times\left(\sum_i \tilde V_i \tilde e_i\right) &= \frac{1}{r}\left(\frac{\partial V_z}{\partial\phi} - \frac{\partial r V_\phi}{\partial z}\right)\tilde e_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\tilde e_\phi \\
&\quad + \frac{1}{r}\left(\frac{\partial r V_\phi}{\partial r} - \frac{\partial V_z}{\partial \phi}\right)\tilde e_z \\
&= \left(\frac{1}{r}\frac{\partial V_z}{\partial\phi} - \frac{\partial V_\phi}{\partial z}\right)\tilde e_r + \left(\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right)\tilde e_\phi \\
&\quad + \left(\frac{\partial V_\phi}{\partial r} - \frac{1}{r}\frac{\partial V_z}{\partial\phi} + \frac{1}{r}V_\phi\right)\tilde e_z \\
\vec\nabla\cdot\left(\sum_i \tilde V_i \tilde e_i\right) &= \frac{1}{r}\left(\frac{\partial (rV_r)}{\partial r} + \frac{\partial V_\phi}{\partial \phi} + \frac{\partial (rV_z)}{\partial z}\right) \\
&= \frac{\partial V_r}{\partial r} + \frac{1}{r}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z} + \frac{1}{r}V_r
\end{aligned}$$

Polar, continued further

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$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\right)\right. \\
&\quad \left. + \frac{\partial}{\partial z}\left(r\frac{\partial\Phi}{\partial z}\right)\right] \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{d^2\Phi}{d\phi^2} + \frac{d^2\Phi}{dz^2}
\end{aligned}$$

Spherical Coordinates

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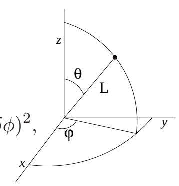
Spherical Coordinates:
r radius, θ polar angle, ϕ azimuth



$$(\delta s)^2 = (\delta r)^2 + r^2(\delta\theta)^2 + r^2\sin^2\theta(\delta\phi)^2,$$

so $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$.

$$\begin{aligned}
\vec\nabla f &= \frac{\partial f}{\partial r}\tilde e_r + \frac{1}{r}\frac{\partial f}{\partial\theta}\tilde e_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\tilde e_\phi \\
\vec\nabla\times\vec V &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}r\sin\theta V_\phi - \frac{\partial}{\partial\phi}rV_\theta\right)\tilde e_r \\
&\quad + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}V_r - \frac{\partial}{\partial r}r\sin\theta V_\phi\right)\tilde e_\theta \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}rV_\theta - \frac{\partial}{\partial\theta}V_r\right)\tilde e_\phi
\end{aligned}$$

Spherical Coordinates, continued

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$$\begin{aligned}
\vec\nabla\times\vec V &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta V_\phi) - \frac{\partial}{\partial\phi}V_\theta\right]\tilde e_r \\
&\quad + \left[\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}V_r - \frac{1}{r}\frac{\partial}{\partial r}(rV_\phi)\right]\tilde e_\theta \\
&\quad + \frac{1}{r}\left[\frac{\partial}{\partial r}(rV_\theta) - \frac{\partial}{\partial\theta}V_r\right]\tilde e_\phi, \\
\vec\nabla\cdot\vec B &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\tilde B_r\right) + \frac{\partial}{\partial\theta}\left(r\sin\theta\tilde B_\theta\right)\right. \\
&\quad \left. + \frac{\partial}{\partial\phi}\left(r\tilde B_\phi\right)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\tilde B_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\tilde B_\theta\right) \\
&\quad + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\tilde B_\phi
\end{aligned}$$

Spherical, continued further

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$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}r^2\sin\theta\frac{\partial\Phi}{\partial r} + \frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta}\right. \\
&\quad \left. + \frac{\partial}{\partial\phi}\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) \\
&\quad + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}.
\end{aligned}$$