Lecture 3

January 28, 2010

- Attenuation in a circular wave guide
- Resonant Cavities

Attenuation in a circular wave guide

Last time we found the modes for a circular wave guide of radius r are given by

$$\begin{aligned}
\psi_{mn}^{\mathrm{TE}} &= J_m(x'_{mn}\rho/r)\cos m\phi, \qquad \text{with} \quad \frac{dJ_m}{dx}(x'_{mn}) = 0, \\
\psi_{mn}^{\mathrm{TM}} &= J_m(x_{mn}\rho/r)\cos m\phi, \qquad \text{with} \quad J_m(x'_{mn}) = 0
\end{aligned}$$

The cutoff wavenumbers and frequencies are
$\gamma_{mn}^{\mathrm{TE}} = x'_{mn}/r$ and $\gamma_{mn}^{\mathrm{TM}} = x_{mn}/r$, with $\omega_\lambda = c\gamma_\lambda$.
We also found for general cylindrical wave guides that the attenuation coefficients are

TM mode: $$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda}\frac{C}{2A}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\xi_\lambda.$$

TE mode: $$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda}\frac{C}{2A}\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\left[\xi_\lambda + \eta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right],$$

The dimensionless quantities ξ_λ, ζ_λ and η_λ are given by

$$\frac{C}{A}\xi_\lambda^{\mathrm{TM}} = \int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Bigg/ \int_A \left|\vec{\nabla}\psi\right|^2,$$

$$\begin{aligned}\frac{C}{A}\xi_\lambda^{\mathrm{TE}} &= \int_\Gamma \left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 \Big/ \gamma_\lambda^2 \int_A |\psi|^2,\\ \frac{C}{A}\zeta_\lambda^{\mathrm{TE}} &= \int_\Gamma |\psi|^2 \Big/ \int_A |\psi|^2,\end{aligned}$$

and $\eta_\lambda^{\mathrm{TE}} = \zeta_\lambda^{\mathrm{TE}} - \xi_\lambda^{\mathrm{TE}}$.

As $\psi(\rho,\phi) = J_m(\gamma\rho)\cos m\phi$.

$$\frac{\partial\psi}{\partial n} = \gamma J_m'(\gamma r)\cos m\phi, \qquad \hat{n}\times\vec{\nabla}_t\psi = \frac{m}{\rho}J_m(\rho)\sin m\phi.$$

The angular integrals are in all case trivial (and even more so if we used the complex modes $e^{-m\phi}$).

For TM:

$$\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 = r\int_0^{2\pi} d\phi\gamma^2 J_m'^2(\gamma r)\cos^2\phi = \pi r\gamma^2 J_m'^2(\gamma r)(1+\delta_{m0}),$$

For TE:

$$\int_\Gamma |\psi|^2 = rJ_m^2(x'_{mn})\int_0^{2\pi}\cos^2 m\phi\, d\phi = \pi r J_m^2(x'_{mn})(1+\delta_{m0}),$$

$$\begin{aligned}\int_\Gamma |\hat{n}\times\nabla_t\psi|^2 &= r\int_0^{2\pi} d\phi\left(\frac{\partial\psi}{r\partial\phi}\right)^2\\ &= \frac{1}{r}J_m^2(x'_{mn})\int_0^{2\pi}(m\sin m\phi)^2\\ &= \frac{\pi m^2}{r}J_m^2(x'_{mn})(1+\delta_{m0}),\end{aligned}$$

For both modes, we need the nontrivial

$$\begin{aligned}\int_A \psi^2 &= \int_0^r \rho d\rho J_m^2(\gamma\rho)\int_0^{2\pi} d\phi\cos^2(m\phi)\\ &= \pi(1+\delta_{m0})\int_0^r \rho d\rho J_m^2(\gamma\rho)\end{aligned}$$

The radial integral

$$\int_0^r \rho d\rho J_m^2(\gamma\rho) = r^2 \int_0^1 u du J_m^2(xu),$$

where x is either x_{mn} (for TM) or x'_{mn} (for TE). The integral is related to the orthonormalization properties of Bessel functions. From Arfken (or "Lecture Notes" → "Notes on Bessel functions") we find

$$\int_0^1 \left[J_m\left(x_{mn}u\right)\right]^2 u du = \frac{1}{2} J_{m+1}^2(x_{mn})$$

$$\int_0^1 \left[J_m\left(x'_{mn}u\right)\right]^2 u du = \frac{1}{2}\left(1 - \frac{m^2}{(x'_{mn})^2}\right) J_m^2(x'_{mn})$$

Thus for the TM modes, we have

$$\begin{aligned}\frac{C}{A}\xi_{mn}^{\mathrm{TM}} &= \int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Bigg/ (\gamma_{mn}^{\mathrm{TM}})^2 \int_A \psi^2 = \frac{\pi r J_m'^{\,2}(x_{mn})}{\frac{\pi r^2}{2} J_{m+1}^2(x_{mn})} \\ &= \frac{2}{r}\frac{J_m'^{\,2}(x_{mn})}{J_{m+1}^2(x_{mn})}\end{aligned}$$

In fact, there is an identity (see footnote again) $J_m'(x) = \frac{m}{x}J_m(x) - J_{m+1}(x)$, which means, as $J_m(x_{mn}) = 0$, that $J_m'(x_{mn}) = -J_{m+1}(x_{mn})$, $\dfrac{C}{A}\xi_{mn}^{\mathrm{TM}} = \dfrac{2}{r}$, and

$$\beta_{mn}^{\mathrm{TM}} = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{r\sigma\delta_\lambda}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\dfrac{\omega_\lambda^2}{\omega^2}}}$$

for all TM modes.

For the TE modes,

$$\begin{aligned}
\frac{C}{A}\xi^{\mathrm{TE}}_{mn} &= \int_\Gamma |\hat{n}\times\nabla_t\psi|^2 \Big/ (\gamma^{\mathrm{TE}}_{mn})^2 \int_A \psi^2 \\
&= \frac{m^2\pi J_m^2(x'_{mn})/r}{\pi(\gamma^{\mathrm{TE}}_{mn})^2 r^2 \frac{1}{2}\left(1-(m/x'_{mn})^2\right) J_m^2(x'_{mn})} \\
&= \frac{2m^2}{r(x'^2_{mn}-m^2)}.
\end{aligned}$$

$$\begin{aligned}
\frac{C}{A}\zeta^{\mathrm{TE}}_{mn} &= \int_\Gamma |\psi|^2 \Big/ \int_A \psi^2 = \frac{\pi r J_m^2(x'_{mn})}{\frac{\pi}{2}\left(1-(m/(x'_{mn})^2)\right) J_m^2(x'_{mn})} \\
&= \frac{2x'^2_{mn}}{r\left(x'^2_{mn}-m^2\right)}.
\end{aligned}$$

So the attenuation coefficient is

$$\beta^{\mathrm{TE}}_{mn} = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{r\sigma\delta_\lambda}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\frac{\omega_\lambda^2}{\omega^2}}}\left[\frac{1}{(x'^2_{mn}-m^2)}+\left(\frac{\omega_\lambda}{\omega}\right)^2\right].$$

For TM modes, $\omega_{mn}^{\rm TM} = x_{mn}c/r$.
For copper, the resistivity is $\rho = \sigma^{-1} = 1.7 \times 10^{-8}\ \Omega \cdot \mathrm{m}$.
Take $\mu_c = \mu_0$. Also $\omega_\lambda = \gamma_\lambda c$. $\delta_\lambda = \sqrt{2/\mu_c \sigma \omega_\lambda}$.
$\epsilon_0 = 8.854 \times 10^{-12}\,\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2$, so

$$\begin{aligned}
\sqrt{\frac{\epsilon}{\mu}}\frac{1}{\sigma\delta_\lambda} &= \sqrt{\frac{c\epsilon_0\gamma_\lambda}{2\sigma}} = 4.75\times 10^{-6}\sqrt{\gamma_\lambda}\sqrt{\frac{\mathrm{m}}{\mathrm{s}}\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}\Omega\mathrm{m}} \\
&= 4.75\times 10^{-6}\ \mathrm{m}^{1/2}\cdot\sqrt{\frac{x_{mn}}{r}}.
\end{aligned}$$

The units combine to $\mathrm{m}^{1/2}$ as $\Omega = \dfrac{\mathrm{V}}{\mathrm{A}} = \dfrac{\mathrm{J/C}}{\mathrm{C/s}} = \mathrm{Nms/C^2}$.
In comparison to the TM_{12} mode for a square of side a, we see that $\beta^{\rm TM} = \frac{a}{2r}\beta_{12}^{\square\mathrm{TM}}$. As the cutoff frequencies are $2.4048c/r$ and $\sqrt{5}\pi c/a$ respectively, we see that the comparable dimensions are $r = (2.4048/\sqrt{5}\pi)a = 0.342a$, much smaller, and then $a/2r = 1.46$, so the smaller pipe does have faster attenuation.

For TE modes, there is an extra factor of

$$\frac{1}{(x'^2_{mn} - m^2)} + \left(\frac{\omega_\lambda}{\omega}\right)^2 .$$

which for the lowest mode is $0.4185 + (\omega_\lambda/\omega)^2$ compared to $0.5 + (\omega_\lambda/\omega)^2$ for the square. But the cutoff frequencies are now $1.841c/r$ and $\sqrt{2}\pi c/a$, so comparable dimensions have $r = 1.841a/\sqrt{2}\pi = 0.414a$.

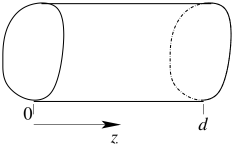
Resonant Cavities

In infinite cylindrical waveguide, have waves with (angular) frequency ω for each arbitrary definite wavenumber k, with $\omega = c\sqrt{k^2 + \gamma_\lambda^2}$. For each mode λ and each $\omega > \omega_\lambda = c\gamma_\lambda$, there are two modes, $k = \pm\sqrt{\omega^2/c^2 - \gamma_\lambda^2}$.

Standing waves by superposition.

Flat conductors at z=0 and z=d.

For TM, the determining field is



$$E_z = \left(\psi^{(k)} e^{ikz} + \psi^{(-k)} e^{-ikz}\right) e^{-i\omega t},$$

$$\vec{E}_t = i\frac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi^{(k)} e^{ikz} + i\frac{-k}{\gamma_\lambda^2}\vec{\nabla}_t\psi^{(-k)} e^{-ikz}$$

$\vec{E}_t = 0$ at endcap so $\psi^{(k)} = \psi^{(-k)}$ (at z=0) and $\sin kd = 0$ (at z=d). So $k = p\pi/d$, $p \in \mathbb{Z}$.

Thus the TM fields are

$$\left.\begin{aligned} E_z &= \cos\left(\frac{p\pi z}{d}\right)\,\psi(x,y) \\ \vec{E}_t &= -\frac{p\pi}{d\gamma_\lambda^2}\sin\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \\ \vec{H}_t &= i\frac{\epsilon\omega}{\gamma_\lambda^2}\cos\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \end{aligned}\right\} \qquad \left\{\begin{array}{c}\text{for TM modes}\\ \text{with } p\in\mathbb{Z}\end{array}\right.$$

Note that in choosing signs we must keep track that half the wave has wavenumber $-k$.

For TE modes, H_z determines all, and must vanish at endcaps (as $\hat{n}\cdot\vec{B}$ vanishes at boundaries). So

$$\left.\begin{aligned} H_z &= \sin\left(\frac{p\pi z}{d}\right)\,\psi(x,y) \\ \vec{H}_t &= \frac{p\pi}{d\gamma_\lambda^2}\cos\left(\frac{p\pi z}{d}\right)\vec{\nabla}_t\psi \\ \vec{E}_t &= -i\frac{\omega\mu}{\gamma_\lambda^2}\sin\left(\frac{p\pi z}{d}\right)\hat{z}\times\vec{\nabla}_t\psi \end{aligned}\right\} \qquad \left\{\begin{array}{c}\text{for TE modes}\\ \text{with } p\in\mathbb{Z}, p\neq 0.\end{array}\right. .$$

Generally the 2D mode λ requires two indices.
For a circular cylinder, we have angular index m,
and radial index n specifying which root of J_m (for TM)
or of $dJ(x)/dx$ (for TE).

$$\gamma_{mn} = \begin{cases} x_{mn}/R & \text{(TM modes)} \quad J_m(x_{mn}) = 0 \\ x'_{mn}/R & \text{(TE modes)} \quad \frac{dJ_m}{dx}(x'_{mn}) = 0 \end{cases}.$$

with R the radius of the cylinder.

Now we have a third index, p.

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x_{mn}^2}{R^2} + \frac{p^2\pi^2}{d^2}} \quad \text{with } p \geq 0 \text{ for TM modes,}$$

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{x'^2_{mn}}{R^2} + \frac{p^2\pi^2}{d^2}} \quad \text{with } p > 0 \text{ for TE modes.}$$

Lowest TM mode, $\omega_{010} = c x_{01}/R = 2.405 c/R$, independent of d.
For TE modes, $p \neq 0$, so lowest mode with $\gamma = x'_{11}/R$ has

$$\omega_{111} = 1.841 \frac{c}{R} \sqrt{1 + 2.912 R^2/d^2}.$$

As this depends on d, such a cavity can be tuned by having a movable piston for one endcap.

Power Loss and Quality Factor

What if conductor not perfect? Power losses in sides and in endcaps. Rate is proportional to $U(t)$, the energy stored inside. Let

$$-\Delta U = \text{energy loss per cycle}, \qquad Q := 2\pi U/|\Delta U| \ .$$

One period is $\Delta t = 2\pi/\omega$. Assume $Q \gg 1$, so $|\Delta U| \ll U$, $\Delta U = -2\pi U/Q = (2\pi/\omega) dU/dt$, so

$$U(t) = U(0)e^{-\omega t/Q}.$$

Q is called the resonance "quality factor" or "Q-value". So if an oscillation excited at time $t = 0$ by momentary external influence,

$$U(t) \propto e^{-\omega t/Q} \Longrightarrow E(t) = E_0 e^{-i\omega_0(1-i/2Q)t}\Theta(t),$$

The Heaviside function $\Theta(t) = 1$ for $t > 0$, $= 0$ for $t < 0$. This $\delta(t)$ excitation consists of equal amounts at all frequencies.

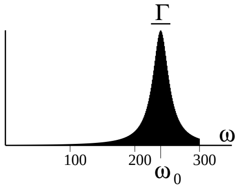
Breit-Wigner

It produces a frequency response

$$\begin{aligned} E(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} E_0 \int_0^{\infty} e^{i(\omega-\omega_0-i\Gamma/2)t} dt \\ &= \frac{iE_0}{\sqrt{2\pi}} \frac{1}{\omega-\omega_0-i\Gamma/2}, \end{aligned}$$

with $\Gamma := \omega_0/Q$.
$|E(\omega)|^2$ gives the response to excitations of any frequency, with

$$|E(\omega)|^2 \propto \frac{1}{(\omega-\omega_0)^2+\Gamma^2/4}.$$



This is called the Breit-Wigner response. Γ is mistakenly called the half-width. Really full-width at half-maximum.

Calculation of power loss as for waveguide, but need to include power loss in endcaps as well. Jackson, pp 373-374. We will skip this.

Earth and Ionosphere:

Not all cavities cylindrical. Consider surface of Earth, and ionosphere, an ionized layer about 100 km up. Concentric conducting spheres acting as endcaps, of a waveguide with no walls, but topology!

Need spherical coordinates, of course. More generally, may need other curvilinear coordinates (as you will for your projects).
So we will digress to discuss curvilinear coordinates.