

Wave velocities

Wave guide traveling modes in z direction

$$\vec{E}, \vec{B} \propto e^{ikz - i\omega t}$$

with dispersion relation $k^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2$.

Same form as for high-frequencies in dielectrics (Jackson 7.61), with $\omega_\lambda \sim$ plasma frequency.

$$\text{Phase velocity } v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \frac{1}{\sqrt{1 - \left(\frac{\omega_\lambda}{\omega}\right)^2}} > \frac{1}{\sqrt{\mu\epsilon}},$$

greater than for unbounded.

$$\text{Group velocity } v_g = \frac{d\omega}{dk} = \frac{1}{\mu\epsilon} \frac{k}{\omega} = \frac{1}{\mu\epsilon} \frac{1}{v_p} < \frac{1}{\sqrt{\mu\epsilon}}, \text{ less than}$$

for the unbounded medium. (I used $kdk = \mu\epsilon\omega d\omega$)

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Last slide assumed perfectly conducting walls. Real walls have energy loss, attenuation, k develops small positive imaginary part $i\beta$ (so extra $e^{-\beta z}$ factor for $\vec{E}$ and for $\vec{H}$). Find β by comparing power lost per unit length to power transmitted.

Power is quadratic in fields. Only real parts of fields are real.

Poynting vector $\vec{S}_{\text{phys}} = \vec{E}_{\text{phys}} \times \vec{H}_{\text{phys}}$ needs

$$\begin{aligned} \vec{E}_{\text{phys}}(x, y, z, t) \\ = \frac{1}{2} \left(\vec{E}(x, y, k, \omega) e^{ikz - i\omega t} + \vec{E}^*(x, y, k, \omega) e^{-ikz + i\omega t} \right). \end{aligned}$$

and similarly for $\vec{H}$.

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So

$$\begin{aligned} \vec{S}_{\text{phys}} &= \vec{E}_{\text{phys}} \times \vec{H}_{\text{phys}} \\ &= \frac{1}{4} \left(\left(\vec{E}(x, y, k, \omega) e^{ikz - i\omega t} + \vec{E}^*(x, y, k, \omega) e^{-ikz + i\omega t} \right) \times \right. \\ &\quad \left. \left(\vec{H}(x, y, k, \omega) e^{ikz - i\omega t} + \vec{H}^*(x, y, k, \omega) e^{-ikz + i\omega t} \right) \right) \\ &= \frac{1}{4} \left(\vec{E}(x, y, k, \omega) \times \vec{H}(x, y, k, \omega) e^{2ikz - 2i\omega t} \right. \\ &\quad + \vec{E}^*(x, y, k, \omega) \times \vec{H}(x, y, k, \omega) \\ &\quad + \vec{E}(x, y, k, \omega) \times \vec{H}^*(x, y, k, \omega) \\ &\quad \left. + \vec{E}^*(x, y, k, \omega) \times \vec{H}^*(x, y, k, \omega) e^{-2ikz + 2i\omega t} \right) \end{aligned}$$

First and last terms rapidly oscillating, average to zero, so

$$\begin{aligned} \langle \vec{S} \rangle &= \frac{1}{4} \left(\vec{E}^*(x, y, k, \omega) \times \vec{H}(x, y, k, \omega) \right. \\ &\quad \left. + \vec{E}(x, y, k, \omega) \times \vec{H}^*(x, y, k, \omega) \right) \\ &= \frac{1}{2} \text{Re} \left(\vec{E}(x, y, k, \omega) \times \vec{H}^*(x, y, k, \omega) \right) \end{aligned}$$

Define the complex $\vec{S} := \frac{1}{2} (\vec{E} \times \vec{H}^*)$, with the physical average flux given by the real part.

Power flow $\propto \int \hat{z} \cdot \text{Re} \vec{S}$, so only the transverse parts of $\vec{E}$ and $\vec{H}$ are needed. Recall

$$\begin{aligned} \text{TM:} \quad E_z &= \psi, \quad \vec{E}_t = i \frac{k}{\gamma_\lambda} \vec{\nabla}_t \psi, \quad \vec{H}_t = i \frac{\epsilon \omega}{\gamma_\lambda^2} \hat{z} \times \vec{\nabla}_t \psi \\ \text{TE:} \quad H_z &= \psi, \quad \vec{H}_t = i \frac{k}{\gamma_\lambda} \vec{\nabla}_t \psi, \quad \vec{E}_t = -i \frac{\mu \omega}{\gamma_\lambda^2} \hat{z} \times \vec{\nabla}_t \psi \end{aligned}$$

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Energy per unit length

$$U = \int_A u = \frac{1}{2} \int_A (\vec{E}_{\text{phys}} \cdot \vec{D}_{\text{phys}} + \vec{B}_{\text{phys}} \cdot \vec{H}_{\text{phys}}),$$

$$\langle U \rangle = \frac{1}{4} \int_A \epsilon |\vec{E}|^2 + \mu |\vec{H}|^2$$

Need z components (ψ or 0) as well as transverse ones. Plugging in is straightforward (see notes), and we find

$$\langle U \rangle = \frac{\omega^2}{2\omega_\lambda^2} \int_A |\psi|^2 \times \begin{cases} \epsilon & \text{TM mode} \\ \mu & \text{TE mode} \end{cases}$$

In either case,

$$\frac{\langle P \rangle}{\langle U \rangle} = \frac{1}{\sqrt{\epsilon \mu}} \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} = v_g.$$

Energy flux = energy density times *group* velocity.

As $\hat{z} \cdot (\vec{\nabla}_t \psi \times \hat{z} \times \vec{\nabla}_t \psi^*) = |\vec{\nabla}_t \psi|^2$, we have

$$P = \hat{z} \cdot \int_A \text{Re} \vec{S} = \frac{\omega k}{2\gamma_\lambda^2} \int_A |\vec{\nabla}_t \psi|^2 \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

The integral

$$\int_A |\vec{\nabla}_t \psi|^2 = \oint_S \psi^* \frac{\partial \psi}{\partial n} - \int_A \psi^* \nabla_t^2 \psi = 0 + \gamma_\lambda^2 \int_A \psi^* \psi.$$

As $\omega_\lambda := \gamma_\lambda / \sqrt{\mu\epsilon}$, $k = \omega \sqrt{\mu\epsilon} \sqrt{1 - \omega_\lambda^2 / \omega^2}$,

$$P = \frac{1}{2\sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_\lambda} \right)^2 \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} \int_A \psi^* \psi \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

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Attenuation and Power Loss

At an interface, we found power loss per unit area is

$$\frac{1}{2\delta\sigma} \left| \vec{H}_{\parallel} \right|^2 = \frac{1}{2\delta\sigma} \left| \hat{n} \times \vec{H} \right|^2,$$

with conductivity σ and skin depth $\delta = \sqrt{2/\mu_c\sigma\omega}$. As the power drops off as the square of the fields, so as $e^{-2\beta z}$

$$\frac{dP}{dz} = -2\beta P(z) = -\frac{1}{2\delta\sigma} \oint_{\Gamma} \left| \hat{n} \times \vec{H} \right|^2 d\ell,$$

where the integral $d\ell$ is over the loop Γ around the interface at fixed z .

β will depend on the mode being considered, so we will call it β_{λ} .

Note resistivity can couple modes, but we will not discuss that.

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Attenuation for TM modes

For a TM mode,

$$\hat{n} \times \vec{H} = \hat{n} \times \vec{H}_t = \frac{i\epsilon\omega}{\gamma_{\lambda}^2} \hat{n} \times (\hat{z} \times \vec{\nabla}_t \psi) = \frac{i\epsilon\omega}{\gamma_{\lambda}^2} (\hat{n} \cdot \vec{\nabla}_t \psi) \hat{z}$$

so

$$\begin{aligned} \beta_{\lambda} &= \frac{1}{4\sigma\delta} \left(\frac{\epsilon\omega}{\gamma_{\lambda}^2} \right)^2 \int_{\Gamma} \left| \frac{\partial\psi}{\partial n} \right|^2 \bigg/ \frac{\omega k \epsilon}{2\gamma_{\lambda}^4} \int_A \left| \vec{\nabla} \psi \right|^2 \\ &= \underbrace{\frac{\omega \epsilon}{2k\sigma\delta} \int_{\Gamma} \left| \frac{\partial\psi}{\partial n} \right|^2 \bigg/ \int_A \left| \vec{\nabla} \psi \right|^2}_{C\xi_{\lambda}/A} \end{aligned}$$

where C is the length of Γ and A the area, and ξ_{λ} is a mode- and geometry-dependent dimensionless number, the average size of the normal derivative to the gradient, which we would expect to be of order 1.

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Attenuation for TE modes

For a TE mode, $\hat{n} \times \vec{H} = \hat{n} \times \vec{H}_t + \hat{n} \times \hat{z} H_z$ so

$$\left| \hat{n} \times \vec{H} \right|^2 = \left| \hat{n} \times \vec{H}_t \right|^2 + |H_z|^2 = \left(\frac{k}{\gamma_{\lambda}^2} \right)^2 \left| \hat{n} \times \vec{\nabla}_t \psi \right|^2 + |\psi|^2.$$

Again let us write

$$\int_{\Gamma} \left| \hat{n} \times \vec{\nabla}_t \psi \right|^2 \bigg/ \int_A \left| \vec{\nabla} \psi \right|^2 = \frac{C}{A} \xi_{\lambda}, \quad \int_{\Gamma} |\psi|^2 \bigg/ \int_A |\psi|^2 = \frac{C}{A} \zeta_{\lambda}.$$

where ζ_{λ} is another dimensionless number of order one, and ξ_{λ} is somewhat differently defined. Then

$$\int_{\Gamma} \left| \hat{n} \times \vec{\nabla}_t \psi \right|^2 \bigg/ \int_A |\psi|^2 = \gamma_{\lambda}^2 \frac{C}{A} \xi_{\lambda}.$$

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Frequency Dependence

The conductivity, permeability and permittivity may be considered approximately frequency-independent, but the skin depth δ goes as $\omega^{-1/2}$, so let us write $\delta = \delta_{\lambda} \sqrt{\omega_{\lambda}/\omega}$. Then we can extract the frequency dependence of the attenuation factors

TM mode:

$$\beta_{\lambda} = \sqrt{\frac{\epsilon}{\mu}} \frac{1}{\sigma \delta_{\lambda}} \frac{C}{2A} \frac{\sqrt{\omega/\omega_{\lambda}}}{\sqrt{1 - \frac{\omega_{\lambda}^2}{\omega^2}}} \xi_{\lambda}.$$

TE mode:

$$\beta_{\lambda} = \sqrt{\frac{\epsilon}{\mu}} \frac{1}{\sigma \delta_{\lambda}} \frac{C}{2A} \frac{\sqrt{\omega/\omega_{\lambda}}}{\sqrt{1 - \frac{\omega_{\lambda}^2}{\omega^2}}} \left[\xi_{\lambda} + \eta_{\lambda} \left(\frac{\omega_{\lambda}}{\omega} \right)^2 \right],$$

where $\eta_{\lambda} = \zeta_{\lambda} - \xi_{\lambda}$.

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SKIP

We will skip section 8.6

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Note that β_{λ} diverges as we approach the cutoff frequency $\omega \rightarrow \omega_{\lambda}$, and $\beta_{\lambda} \sim \sqrt{\omega}$ as $\omega \rightarrow \infty$. Thus there is a minimum, at $\sqrt{3}\omega_{\lambda}$ for TM, and at a geometry-dependent value for TE modes.

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