Wave velocities

Wave guide traveling modes in z direction

$$\vec{E}, \vec{B} \propto e^{ikz-i\omega t}$$

with dispersion relation $k^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2$.
Same form as for high-frequencies in dielectrics (Jackson 7.61), with $\omega_\lambda \sim$ plasma frequency.
Phase velocity $v_p = \dfrac{\omega}{k} = \dfrac{1}{\sqrt{\mu\epsilon}} \dfrac{1}{\sqrt{1-\left(\frac{\omega_\lambda}{\omega}\right)^2}} > \dfrac{1}{\sqrt{\mu\epsilon}}$,
greater than for unbounded.

Group velocity $v_g = \dfrac{d\omega}{dk} = \dfrac{1}{\mu\epsilon}\dfrac{k}{\omega} = \dfrac{1}{\mu\epsilon}\dfrac{1}{v_p} < \dfrac{1}{\sqrt{\mu\epsilon}}$, less than for the unbounded medium. (I used $kdk = \mu\epsilon\omega d\omega$)

Attenuation

Last slide assumed perfectly conducting walls. Real walls have energy lose, attenuation, k develops small positive imaginary part $i\beta$ (so extra $e^{-\beta z}$ factor for $\vec{E}$ and for $\vec{H}$). Find β by comparing power lost per unit length to power transmitted.

Power is quadratic in fields. Only real parts of fields are real.

Poynting vector $\vec{S}_{\text{phys}} = \vec{E}_{\text{phys}} \times \vec{H}_{\text{phys}}$ needs

$$\vec{E}_{\text{phys}}(x, y, z, t) = \frac{1}{2}\left(\vec{E}(x, y, k, \omega)e^{ikz-i\omega t} + \vec{E}^*(x, y, k, \omega)e^{-ikz+i\omega t}\right).$$

and similarly for $\vec{H}$.

So

$$\vec{S}_{\text{phys}} = \vec{E}_{\text{phys}} \times \vec{H}_{\text{phys}}$$
$$= \frac{1}{4}\Bigg(\left(\vec{E}(x,y,k,\omega)e^{ikz-i\omega t} + \vec{E}^*(x,y,k,\omega)e^{-ikz+i\omega t}\right) \times$$
$$\left(\vec{H}(x,y,k,\omega)e^{ikz-i\omega t} + \vec{H}^*(x,y,k,\omega)e^{-ikz+i\omega t}\right)\Bigg)$$
$$= \frac{1}{4}\Bigg(\vec{E}(x,y,k,\omega) \times \vec{H}(x,y,k,\omega)e^{2ikz-2i\omega t}$$
$$+\vec{E}^*(x,y,k,\omega) \times \vec{H}(x,y,k,\omega)$$
$$+\vec{E}(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega)$$
$$+\vec{E}^*(x,y,k,\omega) \times \vec{H}^*(x,y,k,\omega)e^{-2ikz+2i\omega t}\Bigg)$$

First and last terms rapidly oscillating, average to zero, so

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$$\begin{aligned}\langle\vec{S}\rangle &= \frac{1}{4}\Big(\vec{E}^*(x,y,k,\omega)\times\vec{H}(x,y,k,\omega) \\ &\qquad +\vec{E}(x,y,k,\omega)\times\vec{H}^*(x,y,k,\omega)\Big) \\ &= \frac{1}{2}\mathrm{Re}\left(\vec{E}(x,y,k,\omega)\times\vec{H}^*(x,y,k,\omega)\right)\end{aligned}$$

Define the complex $\vec{S} := \frac{1}{2}\left(\vec{E}\times\vec{H}^*\right)$, with the physical average flux given by the real part.
Power flow $\propto \int \hat{z}\cdot\mathrm{Re}\vec{S}$, so only the transverse parts of $\vec{E}$ and $\vec{H}$ are needed. Recall

TM: $\quad E_z = \psi, \quad \vec{E}_t = i\frac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi, \quad \vec{H}_t = i\frac{\epsilon\omega}{\gamma_\lambda^2}\hat{z}\times\vec{\nabla}_t\psi$

TE: $\quad H_z = \psi, \quad \vec{H}_t = i\frac{k}{\gamma_\lambda^2}\vec{\nabla}_t\psi, \quad \vec{E}_t = -i\frac{\mu\omega}{\gamma_\lambda^2}\hat{z}\times\vec{\nabla}_t\psi$

As $\hat{z} \cdot \left(\vec{\nabla}_t \psi \times \hat{z} \times \vec{\nabla}_t \psi^* \right) = \left| \vec{\nabla}_t \psi \right|^2$, we have

$$P = \hat{z} \cdot \int_A \operatorname{Re} S = \frac{\omega k}{2\gamma_\lambda^4} \int_A |\vec{\nabla}_t \psi|^2 \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

The integral

$$\int_A |\vec{\nabla}_t \psi|^2 = \oint_S \psi^* \frac{\partial \psi}{\partial n} - \int_A \psi^* \nabla_t^2 \psi = 0 + \gamma_\lambda^2 \int_A \psi^* \psi.$$

As $\omega_\lambda := \gamma_\lambda / \sqrt{\mu\epsilon}$, $k = \omega \sqrt{\mu\epsilon} \sqrt{1 - \omega_\lambda^2/\omega^2}$,

$$P = \frac{1}{2\sqrt{\mu\epsilon}} \left(\frac{\omega}{\omega_\lambda} \right)^2 \sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} \int_A \psi^* \psi \cdot \begin{cases} \epsilon & \text{(for TM)} \\ \mu & \text{(for TE)} \end{cases}$$

Energy Density

Energy per unit length
$U = \int_A u = \frac{1}{2}\int_A \left(\vec{E}_{\text{phys}} \cdot \vec{D}_{\text{phys}} + \vec{B}_{\text{phys}} \cdot \vec{H}_{\text{phys}}\right)$,

$$\langle U \rangle = \frac{1}{4}\int_A \epsilon|\vec{E}|^2 + \mu|\vec{H}|^2$$

Need z components (ψ or 0) as well as transverse ones. Plugging in is straightforward (see notes), and we find

$$\langle U \rangle = \frac{\omega^2}{2\omega_\lambda^2}\int_A |\psi|^2 \times \begin{cases} \epsilon & \text{TM mode} \\ \mu & \text{TE mode} \end{cases}$$

In either case,

$$\frac{\langle P \rangle}{\langle U \rangle} = \frac{1}{\sqrt{\epsilon\mu}}\sqrt{1 - \frac{\omega_\lambda^2}{\omega^2}} = v_g.$$

Energy flux = energy density times *group* velocity.

Attenuation and Power Loss

At an interface, we found power loss per unit area is

$$\frac{1}{2\delta\sigma}\left|\vec{H}_{\parallel}\right|^2 = \frac{1}{2\delta\sigma}\left|\hat{n}\times\vec{H}\right|^2,$$

with conductivity σ and skin depth $\delta = \sqrt{2/\mu_c\sigma\omega}$. As the power drops off as the square of the fields, so as $e^{-2\beta z}$

$$\frac{dP}{dz} = -2\beta P(z) = -\frac{1}{2\delta\sigma}\oint_\Gamma \left|\hat{n}\times\vec{H}\right|^2 d\ell,$$

where the integral $d\ell$ is over the loop Γ around the interface at fixed z.

β will depend on the mode being considered, so we will call it β_λ.

Note resistivity can couple modes, but we will not discuss that.

Attenuation for TM modes

For a TM mode,

$$\hat{n} \times \vec{H} = \hat{n} \times \vec{H}_t = \frac{i\epsilon\omega}{\gamma_\lambda^2} \hat{n} \times \left(\hat{z} \times \vec{\nabla}_t \psi\right) = \frac{i\epsilon\omega}{\gamma_\lambda^2} \left(\hat{n} \cdot \vec{\nabla}_t \psi\right) \hat{z}$$

so

$$\begin{aligned} \beta_\lambda &= \frac{1}{4\sigma\delta} \left(\frac{\epsilon\omega}{\gamma_\lambda^2}\right)^2 \int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Big/ \frac{\omega k \epsilon}{2\gamma_\lambda^4} \int_A \left|\vec{\nabla}\psi\right|^2 \\ &= \frac{\omega\epsilon}{2k\sigma\delta} \underbrace{\int_\Gamma \left|\frac{\partial\psi}{\partial n}\right|^2 \Big/ \int_A \left|\vec{\nabla}\psi\right|^2}_{C\xi_\lambda/A} \end{aligned}$$

where C is the length of Γ and A the area, and ξ_λ is a mode– and geometry–dependent dimensionless number, the average size of the normal derivative to the gradient, which we would expect to be of order 1.

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Attenuation for TE modes

For a TE mode, $\hat{n}\times\vec{H} = \hat{n}\times\vec{H}_t + \hat{n}\times\hat{z}H_z$ so

$$\left|\hat{n}\times\vec{H}\right|^2 = \left|\hat{n}\times\vec{H}_t\right|^2 + |H_z|^2 = \left(\frac{k}{\gamma_\lambda^2}\right)^2 \left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 + |\psi|^2.$$

Again let us write

$$\int_\Gamma \left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 \Big/ \int_A \left|\vec{\nabla}\psi\right|^2 = \frac{C}{A}\xi_\lambda, \quad \int_\Gamma |\psi|^2 \Big/ \int_A |\psi|^2 = \frac{C}{A}\zeta_\lambda.$$

where ζ_λ is another dimensionless number of order one, and ξ_λ is somewhat differently defined. Then

$$\int_\Gamma \left|\hat{n}\times\vec{\nabla}_t\psi\right|^2 \Big/ \int_A |\psi|^2 = \gamma_\lambda^2\frac{C}{A}\xi_\lambda.$$

Frequency Dependence

The conductivity, permeability and permittivity may be considered approximately frequency-independent, but the skin depth δ goes as $\omega^{-1/2}$, so let us write $\delta = \delta_\lambda\sqrt{\omega_\lambda/\omega}$. Then we can extract the frequency dependence of the attenuation factors

TM mode:

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{\sigma\delta_\lambda}\,\frac{C}{2A}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\dfrac{\omega_\lambda^2}{\omega^2}}}\,\xi_\lambda.$$

TE mode:

$$\beta_\lambda = \sqrt{\frac{\epsilon}{\mu}}\,\frac{1}{\sigma\delta_\lambda}\,\frac{C}{2A}\,\frac{\sqrt{\omega/\omega_\lambda}}{\sqrt{1-\dfrac{\omega_\lambda^2}{\omega^2}}}\left[\xi_\lambda + \eta_\lambda\left(\frac{\omega_\lambda}{\omega}\right)^2\right],$$

where $\eta_\lambda = \zeta_\lambda - \xi_\lambda$.

Note that β_λ diverges as we approach the cutoff frequency $\omega \to \omega_\lambda$,
and $\beta_\lambda \sim \sqrt{\omega}$ as $\omega \to \infty$.
Thus there is a minimum, at $\sqrt{3}\,\omega_\lambda$ for TM, and at a geometry-dependent value for TE modes.

SKIP

We will skip section 8.6