Physics 504, Spring 2010 Electricity and Magnetism

Joel A. Shapiro
shapiro@physics.rutgers.edu

January 21, 2010

Course Information

- Instructor:
  - Joel Shapiro
  - Serin 325
  - 5-5500 X 3886, shapiro@physics
- Book: Jackson: Classical Electrodynamics (3rd Ed.)
- Web home page:
  `www.physics.rutgers.edu/grad/504`
  contains general info, syllabus, lecture and other notes, homework assignments, *etc.*
- Classes: ARC 207, Monday and Thursday, 10:20 (sharp!) - 11:40
- Homework: there will be one or two projects, and homework assignments every week or so. Due dates to be discussed.
- Exams: a midterm and a final.
- Office Hour: Tuesdays, 3:30–4:30, in Serin 325.

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

Course Content

Last term you covered Jackson, Chapters 1—7
We will cover most of chapters 8—16

Everything comes from Maxwell's Equations and the Lorentz Force. We will discuss:

- EM fields confined: waveguides, cavities, optical fibers
- Sources of fields: antennas and their radiation, scattering and diffraction
- Relativity, and relativistic formalism for E&M
- Relativistic particles
- other gauge theories

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

Maxwell's Equations

$$\begin{aligned}
\vec\nabla\cdot\vec E &= \frac{1}{\epsilon_0}\rho_{\rm all} && \text{Gauss for E}\\
\vec\nabla\cdot\vec B &= 0 && \text{Gauss for B}\\
\vec\nabla\times\vec B - \frac{1}{c^2}\frac{\partial\vec E}{\partial t} &= \mu_0\vec J_{\rm all} && \text{Ampère (+Max)}\\
\vec\nabla\times\vec E + \frac{\partial\vec B}{\partial t} &= 0 && \text{Faraday}
\end{aligned}$$

plus the Lorentz force:

$$\vec F = q(\vec E + \vec v\times\vec B)$$

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

In "Ponderable Media"

In the last slide, $\rho_{\rm all}$ and $\vec J_{\rm all}$ represent all charges, both "free" and "induced".
Separate "free" from "induced":

- For the electric field
  - $\vec E$ called *electric field*
  - $\vec P$ called *electric polarization* is induced field
  - $\vec D$ called *electric displacement* is field of "free charges"
  - $\vec D = \epsilon_0\vec E + \vec P$
- For the magnetic field
  - $\vec B$ called *magnetic induction* (unfortunately)
  - $\vec M$ called *magnetization* is the induced field
  - $\vec H$ called *magnetic field*
  - $\vec H = \frac{1}{\mu_0}\vec B - \vec M$

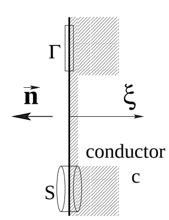
Then the two Maxwell equations with sources, Gauss for $\vec E$ and Ampère, get replaced by

$$\begin{aligned}
\vec\nabla\cdot\vec D &= \rho\\
\vec\nabla\times\vec H - \frac{\partial\vec D}{\partial t} &= \vec J
\end{aligned}$$

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

Interface between conductor and non-conductor

Conductor c: if perfect, no $\vec E$.
Surface charge Σ, eddy currents so no $\vec H$ inside conductor.
Just outside the conductor:
Faraday on loop $\Gamma \longrightarrow E_\parallel \approx 0$
Gauss on pillbox $S \longrightarrow B_\perp \approx 0$



Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

For good (not perfect) conductor, take $\vec{J} = \sigma\vec{E}$ with large *conductivity* σ. Assume time-dependence $\propto e^{-i\omega t}$
Skin depth ξ, $\vec{H}$ varies rapidly.

$$\vec{\nabla}\times\vec{H}_c = \vec{J} + \frac{\partial\vec{D}}{\partial t} \approx \sigma\vec{E},$$

$$\vec{\nabla}\times\vec{E}_c = -\frac{\partial\vec{B}}{\partial t} = i\omega\mu_c H_c$$

Rapid variation with depth ξ dominates, $\vec{\nabla} = -\hat{n}\frac{\partial}{\partial\xi}$, and

$$\vec{E}_c = \frac{1}{\sigma}\vec{J} = -\frac{1}{\sigma}\hat{n}\times\frac{\partial\vec{H}_c}{\partial\xi}, \qquad \vec{H}_c = \frac{i}{\omega\mu_c}\hat{n}\times\frac{\partial\vec{E}_c}{\partial\xi}$$

$$\vec{E}_c = \frac{1}{\sigma}\vec{J} = -\frac{1}{\sigma}\hat{n}\times\frac{\partial\vec{H}_c}{\partial\xi}, \qquad \vec{H}_c = \frac{i}{\omega\mu_c}\hat{n}\times\frac{\partial\vec{E}_c}{\partial\xi}$$

so $\hat{n}\cdot\vec{H}_c = 0$ and

$$\begin{aligned}\hat{n}\times\vec{H}_c &= \frac{i}{\omega\mu_c}\hat{n}\times\left(\hat{n}\times\frac{\partial\vec{E}_c}{\partial\xi}\right)\\ &= -\frac{i}{\sigma\omega\mu_c}\hat{n}\times\left(\hat{n}\times\left[\hat{n}\times\frac{\partial^2\vec{H}_c}{\partial\xi^2}\right]\right)\\ &= \frac{i}{\sigma\omega\mu_c}\frac{\partial^2}{\partial\xi^2}\left(\hat{n}\times\vec{H}_c\right).\end{aligned}$$

Simple DEQ, exponential solution, with $\delta = \sqrt{\dfrac{2}{\mu_c\omega\sigma}}$,

$$\vec{H}_c = \vec{H}_{\|}e^{-\xi/\delta}e^{i\xi/\delta},$$

$H_{\|}$ is tangential field outside surface of conductor.

From $\vec{H}_c = \vec{H}_{\|}e^{-\xi/\delta}e^{i\xi/\delta}$,

$$\vec{E}_c = -\frac{1}{\sigma}\hat{n}\times\frac{\partial\vec{H}_c}{\partial\xi} = \sqrt{\frac{\mu_c\omega}{2\sigma}}(1-i)\hat{n}\times\vec{H}_{\|}e^{-\xi/\delta}e^{i\xi/\delta},$$

which means, by continuity, that just outside the conductor

$$\vec{E}_{\|} = \sqrt{\frac{\mu_c\omega}{2\sigma}}(1-i)\hat{n}\times\vec{H}_{\|}.$$

How much power is dissipated (per unit area?). 2 ways:
1) Flow of energy into conductor: Energy flow given by $\vec{S} = \vec{E}\times\vec{H}$, for real fields $\vec{E}$ and $\vec{H}$.
so[1] $\langle\vec{S}\rangle = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)$,

$$\begin{aligned}\frac{dP_{\rm loss}}{dA} &= -\hat{n}\cdot\langle\vec{S}\rangle\\ &= -\frac{1}{2}\sqrt{\frac{\mu_c\omega}{2\sigma}}\hat{n}\cdot\mathrm{Re}\left[(1-i)(\hat{n}\times\vec{H}_{\|})\times\vec{H}_{\|}^*\right]\\ &= \frac{\mu_c\omega\delta}{4}|\vec{H}_{\|}|^2 = \frac{1}{2\sigma\delta}|\vec{H}_{\|}|^2\end{aligned}$$

Method 2, Ohmic heating, power lost per unit volume $\frac{1}{2}\vec{J}\cdot\vec{E}^* = |\vec{J}|^2/2\sigma$, $|\vec{J}| = \sigma\vec{E}_c = \frac{\sqrt{2}}{\delta}|\vec{H}_{\|}|e^{-\xi/\delta}$, the power loss per unit area is

$$\frac{dP_{\rm loss}}{dA} = = \frac{1}{\delta^2\sigma}|\vec{H}_{\|}|^2\int_0^\infty d\xi\, e^{-2\xi/\delta} = \frac{1}{2\delta\sigma}|\vec{H}_{\|}|^2.$$

Agrees with method 1.

[1] The $\frac{1}{2}$, Re, and $*$ will be discussed in lecture 3.

In terms of surface current

$$\begin{aligned}\vec{K}_{\rm eff} &= \int_0^\infty d\xi\,\vec{J}(\xi) = \frac{1}{\delta}\hat{n}\times\vec{H}_{\|}\int_0^\infty d\xi\,(1-i)e^{-\xi(1-i)/\delta}\\ &= \hat{n}\times\vec{H}_{\|}.\end{aligned}$$

Thus

$$\frac{dP_{\rm loss}}{dA} = \frac{1}{2\sigma\delta}|\vec{K}_{\rm eff}|^2.$$

$\dfrac{1}{\sigma\delta}$ is surface resistance (per unit area) and $\dfrac{\vec{E}_{\|}}{\vec{K}_{\rm eff}} = \dfrac{1-i}{\sigma\delta}$ is the surface impediance Z.

Wave Guides

For electromagnetic fields with a fixed geometry of linear materials, fourier transform decouples, and we can work with frequency modes,

$$\begin{aligned}\vec{E}(\vec{x},t) &= \vec{E}(x,y,z)\,e^{-i\omega t}\\ \vec{B}(\vec{x},t) &= \vec{B}(x,y,z)\,e^{-i\omega t}\end{aligned}$$

Actually the fields are the real parts of these complex expressions.
If $\rho = 0$, $\vec{J} = 0$, Maxwell gives

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B}, \qquad \vec{\nabla}\cdot\vec{E} = 0, \qquad \vec{\nabla}\cdot\vec{B} = 0,$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{\nabla}\times\vec{H} = \mu\frac{\partial\vec{D}}{\partial t} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} = -i\omega\mu\epsilon\vec{E}.$$

Then

$$\nabla^2\vec{E} = -\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) = -\vec{\nabla}\times(i\omega\vec{B}) = -\omega^2\mu\epsilon\vec{E}.$$

and similarly for $\vec{B}$, so we get Helmholtz equations

$$\left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{E} = 0, \qquad \left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{B} = 0.$$

Consider a waveguide, a cylinder of arbitrary cross section but uniform in z. Fourier transform in z

$$\begin{aligned} \vec{E}(x,y,z,t) &= \vec{E}(x,y)e^{ikz-i\omega t} \\ \vec{B}(x,y,z,t) &= \vec{B}(x,y)e^{ikz-i\omega t} \end{aligned}$$

k can take either sign (and a standing wave is a superposition of $k = \pm|k|$). The Helmholtz equations give

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right]\begin{pmatrix}\vec{E}(x,y)\\ \vec{B}(x,y)\end{pmatrix} = 0, \quad \nabla_t^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Decompose longitudinal and transverse

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro: Course Info, Interface, Power Loss
Wave Guides: TEM, TE, and TM Modes, TEM modes, Example

Let

$$\begin{aligned} \vec{E} &= E_z\hat{z} + \vec{E}_t \\ \vec{B} &= B_z\hat{z} + \vec{B}_t \end{aligned} \quad \text{with} \quad \begin{aligned} \vec{E}_t &\perp \hat{z} \\ \vec{B}_t &\perp \hat{z} \end{aligned}$$

$$\begin{aligned} (\vec{\nabla}\times\vec{E})_z &= (\vec{\nabla}_t\times\vec{E}_t)_z = i\omega B_z, \\ (\vec{\nabla}\times\vec{E})_\perp &= \hat{z}\times\frac{\partial\vec{E}_t}{\partial z} - \hat{z}\times\nabla_t E_z = i\omega\vec{B}_t. \end{aligned}$$

For any vector $\vec{V}$, $\hat{z}\times(\hat{z}\times\vec{V}) = -\vec{V} + \hat{z}(\hat{z}\cdot V)$, so for a transverse vector $\hat{z}\times(\hat{z}\times\vec{V}_t) = -\vec{V}_t$. Taking $\hat{z}\,\times$ last equation,

$$\frac{\partial\vec{E}_t}{\partial z} - \vec{\nabla}_t E_z = -i\omega\hat{z}\times\vec{B}_t. \tag{1}$$

Similarly decomposition of $\vec{\nabla}\times\vec{B} = -i\omega\mu\epsilon\vec{E}$ gives

$$\begin{aligned} \left(\vec{\nabla}_t\times\vec{B}_t\right)_z &= -i\omega\mu\epsilon E_z \\ \frac{\partial\vec{B}_t}{\partial z} - \vec{\nabla}_t B_z &= i\omega\mu\epsilon\hat{z}\times\vec{E}_t. \end{aligned} \tag{2}$$

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro: Course Info, Interface, Power Loss
Wave Guides: TEM, TE, and TM Modes, TEM modes, Example

Divergencelessness:

$$\vec{\nabla}_t\cdot\vec{E}_t + \frac{\partial E_z}{\partial z} = 0, \qquad \vec{\nabla}_t\cdot\vec{B}_t + \frac{\partial B_z}{\partial z} = 0.$$

Equations (1) and (2), with the fourier transform in z, give

$$\begin{aligned} ik\vec{E}_t + i\omega\hat{z}\times\vec{B}_t &= \vec{\nabla}_t E_z & (3)\\ ik\vec{B}_t - i\omega\mu\epsilon\hat{z}\times\vec{E}_t &= \vec{\nabla}_t B_z & (4) \end{aligned}$$

Solving 4 for $\vec{B}_t$ and plugging into 3, and then the reverse for $\vec{E}_t$, give

$$\begin{aligned} E_t &= i\frac{k\vec{\nabla}_t E_z - \omega\hat{z}\times\vec{\nabla}_t B_z}{\omega^2\mu\epsilon - k^2} & (5)\\ B_t &= i\frac{k\vec{\nabla}_t B_z + \omega\mu\epsilon\hat{z}\times\vec{\nabla}_t E_z}{\omega^2\mu\epsilon - k^2} & (6) \end{aligned}$$

Unless $k^2 = k_0^2 := \mu\epsilon\omega^2$, E_z and B_z determine the rest.

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro: Course Info, Interface, Power Loss
Wave Guides: TEM, TE, and TM Modes, TEM modes, Example

We have seen that E_z and B_z largely determine the fields, and these satisfy the two-dimensional Helmholtz equation

$$\left(\nabla_t^2 + \gamma^2\right)\psi = 0 \qquad \text{with} \quad \gamma^2 = \mu\epsilon\omega^2 - k^2 \tag{7}$$

If the walls of the waveguide are very good conductors, we may impose the perfect conductor conditions $E_\parallel \approx 0$ and $B_\perp \approx 0$ on the boundary S of the two-dimensional cross section. E_z is parallel to the boundary so $E_z|_S = 0$. Also the component of $\vec{E}_t$ parallel to the boundary vanishes at the wall, so $\vec{E}_t$ is in the $\pm\hat{n}$ direction. Then from the $\hat{n}$ component of (2) (normal to the boundary)

$$\frac{\partial\hat{n}\cdot\vec{B}_t}{\partial z} - \hat{n}\cdot\vec{\nabla}_t B_z = i\omega\mu\epsilon\hat{n}\cdot\left(\hat{z}\times\vec{E}_t\right) \Longrightarrow 0 - \frac{\partial B_z}{\partial n} = 0,$$

where $\partial/\partial n$ is the derivative normal to the surface. So we have Dirichlet conditions on E_z and Neumann conditions for B_z.

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro: Course Info, Interface, Power Loss
Wave Guides: TEM, TE, and TM Modes, TEM modes, Example

In general, nonzero solutions exist only for discrete values of γ, and those values are generally different for Dirichlet and for Neumann. So we need to consider

- TEM modes, with $E_z(x,y) = B_z(x,y) \equiv 0$. That is, there are no longitudinal fields, both electric (E) and magnetic (M) fields are purely transverse to the direction z of propagation.
- TE modes, $E_z(x,y) \equiv 0$, and the transverse fields are determined by the gradiant of $B_z = \psi$, a solution of (7) with Neumann conditions.
- TM modes, $B_z(x,y) \equiv 0$, and the transverse fields are determined by $E_z = \psi$, a solution of (7) with zero boundary conditions.

TEM modes

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro: Course Info, Interface, Power Loss
Wave Guides: TEM, TE, and TM Modes, TEM modes, Example

With $E_z(x,y) = B_z(x,y) \equiv 0$, (5) and (6) $\Longrightarrow$ everything vanishes or the denominator vanishes,

$$k = \pm k_0 \qquad \text{with} \quad k_0 = \sqrt{\mu\epsilon}\,\omega$$

Wave travels $\parallel z$ with speed $1/\sqrt{\mu\epsilon}$, same as for infinite medium. No dispersion.
$\vec{\nabla}_t\cdot\vec{E}_t = 0$ and $\vec{\nabla}_t\times\vec{E}_t = i\omega B_z = 0$, so $\exists\Phi \ni \vec{E}_t = -\vec{\nabla}_t\Phi$ (though Φ might not be single valued) and $\nabla^2\Phi = 0$.
As $\vec{E}_\parallel\Big|_S = 0$, Φ = constant on each boundary. If cross section simply connected, Φ = constant, $\vec{E} = 0$
No TEM modes on simply connected cylinder
Yes TEM modes on coaxial cable, or two parallel wires.

TE and TM modes

Equations (5) and (6) simplify for

- TM modes, $B_z = 0$, $\gamma^2 \vec{E}_t = ik\vec{\nabla}_t E_z$,
  $\gamma^2 \vec{B}_t = i\mu\epsilon\omega \hat{z} \times \vec{\nabla}_t E_z$, so $\vec{H}_t = \epsilon\omega k^{-1}\hat{z} \times \vec{E}_t$.
- TE modes, $E_z = 0$, $\gamma^2 \vec{B}_t = ik\vec{\nabla}_t B_z$,
  $\gamma^2 \vec{E}_t = -i\omega \hat{z} \times \vec{\nabla}_t B_z$, so

$$\vec{E}_t = -\omega\hat{z} \times \vec{B}_t/k \underset{\hat{z}\times}{\Longrightarrow} H_t = k\hat{z} \times E_t/\mu\omega.$$

In either case, $\vec{H}_t = \frac{1}{Z}\hat{z} \times \vec{E}_t$, with

$$Z = \begin{cases} k/\epsilon\omega = (k/k_0)\sqrt{\mu/\epsilon} & \text{TM} \\ \mu\omega/k = (k_0/k)\sqrt{\mu/\epsilon} & \text{TE} \end{cases}$$

To Summarize

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

Solutions given by $\psi(x, y)$, with $\left(\nabla_t^2 + \gamma^2\right)\psi = 0$, $\gamma^2 = \mu\epsilon\omega^2 - k^2$, by

$$\begin{array}{lll} \text{TM:} & E_z = \psi e^{ikz-i\omega t}, & \vec{E}_t = ik\gamma^{-2}\vec{\nabla}_t\psi e^{ikz-i\omega t} \\ & \text{with} \quad \psi|_\Gamma = 0 & \\ \text{TE:} & H_z = \psi e^{ikz-i\omega t}, & \vec{H}_t = ik\gamma^{-2}\vec{\nabla}_t\psi e^{ikz-i\omega t} \\ & \text{with} \quad \hat{n}\cdot\vec{\nabla}_t\psi|_\Gamma = 0 & \end{array}$$

By looking at $0 = \int_A \psi^* \left(\nabla_t^2 + \gamma^2\right)\psi$ we can show $\gamma^2 \geq 0$. There are solutions for **discrete** values γ_λ, so only certain wave numbers k_λ for a given frequency can propagate:

$$k_\lambda^2 = \mu\epsilon\omega^2 - \gamma_\lambda^2,$$

and only frequencies $\omega > \omega_\lambda := \gamma_\lambda/\sqrt{\mu\epsilon}$ can propagate, and $k_\lambda < \sqrt{\mu\epsilon}\,\omega$, the infinite medium wavenumber. Phase velocity $v_p = \omega/k_\lambda$ is greater than in the infinite medium.

Example: Circular Wave Guide

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

Jackson does rectangle. You should too. Needed to do homework.

We will consider a circular pipe of (inner) radius r. Of course we should use polar coordinates ρ, ϕ, with

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}, \qquad \text{try} \quad \psi(\rho,\phi) = R(\rho)\Phi(\phi),$$

$$\left(\nabla_t^2 + \gamma^2\right)\psi = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2 R(\rho)\right)\Phi(\phi) + \frac{1}{\rho^2}R(\rho)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

Divide by $R(\rho)\Phi(\phi)$ and multiply by ρ^2:

$$\frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) + \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

Example: Circular Wave Guide

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

Jackson does rectangle. You should too. Needed to do homework.

We will consider a circular pipe of (inner) radius r. Of course we should use polar coordinates ρ, ϕ, with

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}, \qquad \text{try} \quad \psi(\rho,\phi) = R(\rho)\Phi(\phi),$$

$$\left(\nabla_t^2 + \gamma^2\right)\psi = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2 R(\rho)\right)\Phi(\phi) + \frac{1}{\rho^2}R(\rho)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

Divide by $R(\rho)\Phi(\phi)$ and multiply by ρ^2:

$$\begin{aligned} \frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) &= C \\ \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} &= -C. \end{aligned}$$

Solving it

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

Φ first:

$$\frac{\partial^2\Phi(\phi)}{\partial\phi^2} + C\Phi(\phi) = 0$$

$\Phi(\phi) = e^{\pm i\sqrt{C}\phi}$. Periodicity $\Longrightarrow \sqrt{C} = m \in \mathbb{Z}$.

Now $R(\rho)$:

$$\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \gamma^2\rho^2 - m^2\right)R(\rho) = 0$$

Bessel equation, solutions regular at origin are

$$R(\rho) \propto J_m(\gamma\rho), \quad \text{so} \quad \psi(\rho,\phi) = \sum_{m,n} A_{m,n}J_m(\gamma_{mn}\rho)e^{im\phi}.$$

γ_{mn} is determined by boundary conditions...

Physics 504, Spring 2010 Electricity and Magnetism
Shapiro
Intro
Course Info
Interface
Power Loss
Wave Guides
TEM, TE, and TM Modes
TEM modes
Example

Boundary conditions:

For TM, $\psi(r,\phi) = 0 \Longrightarrow J_m(\gamma r) = 0$, so $\gamma_{mn}^{\text{TM}} = x_{mn}/r$ where x_{mn} is the n'th value of $x > 0$ for which $J_m(x) = 0$, given on page 114.

For TE, $\hat{n}\cdot\vec{\nabla}_t\psi(r,\phi) = 0 \Longrightarrow \frac{dJ_m}{dr}(\gamma r) = 0$, so $\gamma_{mn}^{\text{TM}} = x'_{mn}/r$ where x'_{mn} is the n'th value of $x > 0$ for which $dJ_m(x)/dx = 0$, given on page 370.

Thus the lowest cutoff frequency is the $m = 1$ TE mode, with $x'_{11} = 1.841$ while the lowest TM mode or circularly symmetric mode has $x_{01} = 2.405$.

For a waveguide 5 cm in diameter, with air or vacuum inside, the cutoff frequencies are $f = \frac{\omega}{2\pi} = 3.5$ GHz for the lowest TE and 4.6 GHz for the lowest TM modes.