Physics 504, Spring 2010 Electricity and Magnetism

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January 21, 2010

Course Information

- Instructor:
  - Joel Shapiro
  - Serin 325
  - 5-5500 X 3886, shapiro@physics
- Book: Jackson: Classical Electrodynamics (3rd Ed.)
- Web home page:
  `www.physics.rutgers.edu/grad/504`
  contains general info, syllabus, lecture and other notes, homework assignments, *etc.*
- Classes: ARC 207, Monday and Thursday,
  10:20 (sharp!) - 11:40
- Homework: there will be one or two projects, and homework assignments every week or so. Due dates to be discussed.
- Exams: a midterm and a final.
- Office Hour: Tuesdays, 3:30–4:30, in Serin 325.

Course Content

Last term you covered Jackson, Chapters 1—7
We will cover most of chapters 8—16

Everything comes from Maxwell's Equations and the Lorentz Force. We will discuss:

- EM fields confined: waveguides, cavities, optical fibers
- Sources of fields: antennas and their radiation, scattering and diffraction
- Relativity, and relativistic formalism for E&M
- Relativistic particles
- other gauge theories

Maxwell's Equations

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E} &= \frac{1}{\epsilon_0}\rho_{\text{all}} && \text{Gauss for E} \\
\vec{\nabla} \cdot \vec{B} &= 0 && \text{Gauss for B} \\
\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} &= \mu_0 \vec{J}_{\text{all}} && \text{Ampère (+Max)} \\
\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0 && \text{Faraday}
\end{aligned}$$

plus the Lorentz force:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

In "Ponderable Media"

In the last slide, ρ_{all} and $\vec{J}_{\text{all}}$ represent all charges, both "free" and "induced".

Separate "free" from "induced":

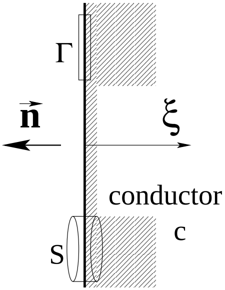
- For the electric field
  - $\vec{E}$ called *electric field*
  - $\vec{P}$ called *electric polarization* is induced field
  - $\vec{D}$ called *electric displacement* is field of "free charges"
  - $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$
- For the magnetic field
  - $\vec{B}$ called *magnetic induction* (unfortunately)
  - $\vec{M}$ called *magnetization* is the induced field
  - $\vec{H}$ called *magnetic field*
  - $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$

Then the two Maxwell equations with sources, Gauss for $\vec{E}$ and Ampère, get replaced by

$$\begin{aligned} \vec{\nabla} \cdot \vec{D} &= \rho \\ \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} &= \vec{J} \end{aligned}$$

Interface between conductor and non-conductor

Conductor c: if perfect, no $\vec{E}$.
Surface charge Σ, eddy currents so no $\vec{H}$ inside conductor.
Just outside the conductor:
Faraday on loop $\Gamma \longrightarrow E_{\|} \approx 0$
Gauss on pillbox $S \longrightarrow B_{\perp} \approx 0$



For good (not perfect) conductor, take $\vec{J} = \sigma \vec{E}$ with large *conductivity* σ. Assume time-dependence $\propto e^{-i\omega t}$
Skin depth ξ, $\vec{H}$ varies rapidly.

$$\vec{\nabla} \times \vec{H}_c = \vec{J} + \frac{\partial \vec{D}}{\partial t} \approx \sigma \vec{E},$$

$$\vec{\nabla} \times \vec{E}_c = -\frac{\partial \vec{B}}{\partial t} = i\omega\mu_c H_c$$

Rapid variation with depth ξ dominates, $\vec{\nabla} = -\hat{n}\frac{\partial}{\partial \xi}$, and

$$\vec{E}_c = \frac{1}{\sigma}\vec{J} = -\frac{1}{\sigma}\hat{n} \times \frac{\partial \vec{H}_c}{\partial \xi}, \qquad \vec{H}_c = \frac{i}{\omega\mu_c}\hat{n} \times \frac{\partial \vec{E}_c}{\partial \xi}$$

$$\vec{E}_c = \frac{1}{\sigma}\vec{J} = -\frac{1}{\sigma}\hat{n} \times \frac{\partial \vec{H}_c}{\partial \xi}, \qquad \vec{H}_c = \frac{i}{\omega\mu_c}\hat{n} \times \frac{\partial \vec{E}_c}{\partial \xi}$$

so $\hat{n} \cdot \vec{H}_c = 0$ and

$$\begin{aligned} \hat{n} \times \vec{H}_c &= \frac{i}{\omega\mu_c}\hat{n} \times \left(\hat{n} \times \frac{\partial \vec{E}_c}{\partial \xi}\right) \\ &= -\frac{i}{\sigma\omega\mu_c}\hat{n} \times \left(\hat{n} \times \left[\hat{n} \times \frac{\partial^2 \vec{H}_c}{\partial \xi^2}\right]\right) \\ &= \frac{i}{\sigma\omega\mu_c}\frac{\partial^2}{\partial \xi^2}\left(\hat{n} \times \vec{H}_c\right). \end{aligned}$$

Simple DEQ, exponential solution, with $\delta = \sqrt{\frac{2}{\mu_c\omega\sigma}}$,

$$\vec{H}_c = \vec{H}_\parallel e^{-\xi/\delta} e^{i\xi/\delta},$$

$H_\parallel$ is tangential field outside surface of conductor.

From $\vec{H}_c = \vec{H}_\parallel e^{-\xi/\delta} e^{i\xi/\delta}$,

$$\vec{E}_c = -\frac{1}{\sigma}\hat{n} \times \frac{\partial \vec{H}_c}{\partial \xi} = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)\hat{n} \times \vec{H}_\parallel e^{-\xi/\delta} e^{i\xi/\delta},$$

which means, by continuity, that just outside the conductor

$$\vec{E}_\parallel = \sqrt{\frac{\mu_c \omega}{2\sigma}}(1-i)\hat{n} \times \vec{H}_\parallel.$$

How much power is dissipated (per unit area?). 2 ways:
1) Flow of energy into conductor: Energy flow given by $\vec{S} = \vec{E} \times \vec{H}$, for real fields $\vec{E}$ and $\vec{H}$.
so[1] $\langle \vec{S} \rangle = \frac{1}{2}\mathrm{Re}\left(\vec{E} \times \vec{H}^*\right)$,

$$\begin{aligned}
\frac{dP_{\text{loss}}}{dA} &= -\hat{n} \cdot \langle \vec{S} \rangle \\
&= -\frac{1}{2}\sqrt{\frac{\mu_c \omega}{2\sigma}}\hat{n} \cdot \mathrm{Re}\left[(1-i)(\hat{n} \times \vec{H}_\parallel) \times \vec{H}_\parallel^*\right] \\
&= \frac{\mu_c \omega \delta}{4}|\vec{H}_\parallel|^2 = \frac{1}{2\sigma\delta}|\vec{H}_\parallel|^2
\end{aligned}$$

Method 2, Ohmic heating, power lost per unit volume $\frac{1}{2}\vec{J} \cdot \vec{E}^* = |\vec{J}|^2/2\sigma$, $|\vec{J}| = \sigma \vec{E}_c = \frac{\sqrt{2}}{\delta}|\vec{H}_\parallel|e^{-\xi/\delta}$, the power loss per unit area is

$$\frac{dP_{\text{loss}}}{dA} = = \frac{1}{\delta^2\sigma}|\vec{H}_\parallel|^2 \int_0^\infty d\xi\, e^{-2\xi/\delta} = \frac{1}{2\delta\sigma}|\vec{H}_\parallel|^2.$$

Agrees with method 1.

[1] The $\frac{1}{2}$, Re, and $*$ will be discussed in lecture 3.

In terms of surface current

$$\begin{aligned}\vec{K}_{\text{eff}} &= \int_0^\infty d\xi\, \vec{J}(\xi) = \frac{1}{\delta}\hat{n}\times\vec{H}_\parallel \int_0^\infty d\xi\,(1-i)e^{-\xi(1-i)/\delta} \\ &= \hat{n}\times\vec{H}_\parallel.\end{aligned}$$

Thus

$$\frac{dP_{\text{loss}}}{dA} = \frac{1}{2\sigma\delta}|\vec{K}_{\text{eff}}|^2.$$

$\frac{1}{\sigma\delta}$ is surface resistance (per unit area) and $\frac{\vec{E}_\parallel}{\vec{K}_{\text{eff}}} = \frac{1-i}{\sigma\delta}$ is the surface impedance Z.

Wave Guides

For electromagnetic fields with a fixed geometry of linear materials, fourier transform decouples, and we can work with frequency modes,

$$\begin{aligned}\vec{E}(\vec{x},t) &= \vec{E}(x,y,z)\; e^{-i\omega t}\\ \vec{B}(\vec{x},t) &= \vec{B}(x,y,z)\; e^{-i\omega t}\end{aligned}$$

Actually the fields are the real parts of these complex expressions.
If $\rho = 0$, $\vec{J} = 0$, Maxwell gives

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} = i\omega\vec{B}, \qquad \vec{\nabla}\cdot\vec{E} = 0, \qquad \vec{\nabla}\cdot\vec{B} = 0,$$

$$\vec{\nabla}\times\vec{B} = \mu\vec{\nabla}\times\vec{H} = \mu\frac{\partial\vec{D}}{\partial t} = \mu\epsilon\frac{\partial\vec{E}}{\partial t} = -i\omega\mu\epsilon\vec{E}.$$

Then

$$\nabla^2\vec{E} = -\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) + \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) = -\vec{\nabla}\times(i\omega\vec{B}) = -\omega^2\mu\epsilon\vec{E}.$$

and similarly for $\vec{B}$, so we get Helmholtz equations

$$\left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{E} = 0, \qquad \left(\nabla^2 + \omega^2\mu\epsilon\right)\vec{B} = 0.$$

Consider a waveguide, a cylinder of arbitrary cross section but uniform in z. Fourier transform in z

$$\begin{aligned}\vec{E}(x,y,z,t) &= \vec{E}(x,y)e^{ikz-i\omega t}\\ \vec{B}(x,y,z,t) &= \vec{B}(x,y)e^{ikz-i\omega t}\end{aligned}$$

k can take either sign (and a standing wave is a superposition of $k = \pm|k|$). The Helmholtz equations give

$$\left[\nabla_t^2 + (\mu\epsilon\omega^2 - k^2)\right]\begin{pmatrix}\vec{E}(x,y)\\ \vec{B}(x,y)\end{pmatrix} = 0, \quad \nabla_t^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

Decompose longitudinal and transverse

Let

$$\vec{E} = E_z\hat{z} + \vec{E}_t \qquad \text{with} \qquad \vec{E}_t \perp \hat{z}$$
$$\vec{B} = B_z\hat{z} + \vec{B}_t \qquad\qquad\qquad \vec{B}_t \perp \hat{z}$$

$$\begin{aligned}(\vec{\nabla} \times \vec{E})_z &= (\vec{\nabla}_t \times \vec{E}_t)_z = i\omega B_z, \\ (\vec{\nabla} \times \vec{E})_\perp &= \hat{z} \times \frac{\partial \vec{E}_t}{\partial z} - \hat{z} \times \nabla_t E_z = i\omega \vec{B}_t.\end{aligned}$$

For any vector $\vec{V}$, $\hat{z} \times (\hat{z} \times \vec{V}) = -\vec{V} + \hat{z}(\hat{z} \cdot V)$, so for a transverse vector $\hat{z} \times (\hat{z} \times \vec{V}_t) = -\vec{V}_t$. Taking $\hat{z} \times$ last equation,

$$\frac{\partial \vec{E}_t}{\partial z} - \vec{\nabla}_t E_z = -i\omega \hat{z} \times \vec{B}_t. \tag{1}$$

Similarly decomposition of $\vec{\nabla} \times \vec{B} = -i\omega\mu\epsilon\vec{E}$ gives

$$\begin{aligned}\left(\vec{\nabla}_t \times \vec{B}_t\right)_z &= -i\omega\mu\epsilon E_z \\ \frac{\partial \vec{B}_t}{\partial z} - \vec{\nabla}_t B_z &= i\omega\mu\epsilon \hat{z} \times \vec{E}_t. \qquad (2)\end{aligned}$$

Divergencelessness:

$$\vec{\nabla}_t \cdot \vec{E}_t + \frac{\partial E_z}{\partial z} = 0, \qquad \vec{\nabla}_t \cdot \vec{B}_t + \frac{\partial B_z}{\partial z} = 0.$$

Equations (1) and (2), with the fourier transform in z, give

$$ik\vec{E}_t + i\omega\hat{z} \times \vec{B}_t = \vec{\nabla}_t E_z \tag{3}$$
$$ik\vec{B}_t - i\omega\mu\epsilon\hat{z} \times \vec{E}_t = \vec{\nabla}_t B_z \tag{4}$$

Solving 4 for $\vec{B}_t$ and plugging into 3, and then the reverse for $\vec{E}_t$, give

$$E_t = i\frac{k\vec{\nabla}_t E_z - \omega\hat{z} \times \vec{\nabla}_t B_z}{\omega^2\mu\epsilon - k^2} \tag{5}$$

$$B_t = i\frac{k\vec{\nabla}_t B_z + \omega\mu\epsilon\hat{z} \times \vec{\nabla}_t E_z}{\omega^2\mu\epsilon - k^2} \tag{6}$$

Unless $k^2 = k_0^2 := \mu\epsilon\omega^2$, E_z and B_z determine the rest.

We have seen that E_z and B_z largely determine the fields, and these satisfy the two-dimensional Helmholtz equation

$$\left(\nabla_t^2 + \gamma^2\right)\psi = 0 \qquad \text{with} \quad \gamma^2 = \mu\epsilon\omega^2 - k^2 \qquad (7)$$

If the walls of the waveguide are very good conductors, we may impose the perfect conductor conditions $E_\parallel \approx 0$ and $B_\perp \approx 0$ on the boundary S of the two-dimensional cross section. E_z is parallel to the boundary so $E_z|_S = 0$. Also the component of $\vec{E}_t$ parallel to the boundary vanishes at the wall, so $\vec{E}_t$ is in the $\pm\hat{n}$ direction. Then from the $\hat{n}$ component of (2) (normal to the boundary)

$$\frac{\partial \hat{n}\cdot\vec{B}_t}{\partial z} - \hat{n}\cdot\vec{\nabla}_t B_z = i\omega\mu\epsilon\hat{n}\cdot\left(\hat{z}\times\vec{E}_t\right) \Longrightarrow 0 - \frac{\partial B_z}{\partial n} = 0,$$

where $\partial/\partial n$ is the derivative normal to the surface. So we have Dirichlet conditions on E_z and Neumann conditions for B_z.

In general, nonzero solutions exist only for discrete values of γ, and those values are generally different for Dirichlet and for Neumann. So we need to consider

- TEM modes, with $E_z(x, y) = B_z(x, y) \equiv 0$. That is, there are no longitudinal fields, both electric (E) and magnetic (M) fields are purely transverse to the direction z of propagation.
- TE modes, $E_z(x, y) \equiv 0$, and the transverse fields are determined by the gradiant of $B_z = \psi$, a solution of (7) with Neumann conditions.
- TM modes, $B_z(x, y) \equiv 0$, and the transverse fields are determined by $E_z = \psi$, a solution of (7) with zero boundary conditions.

TEM modes

With $E_z(x,y) = B_z(x,y) \equiv 0$, (5) and (6) $\Longrightarrow$ everything vanishes or the denominator vanishes,

$$k = \pm k_0 \qquad \text{with} \quad k_0 = \sqrt{\mu\epsilon}\,\omega$$

Wave travels $\parallel z$ with speed $1/\sqrt{\mu\epsilon}$, same as for infinite medium. No dispersion.
$\vec{\nabla}_t \cdot \vec{E}_t = 0$ and $\vec{\nabla}_t \times \vec{E}_t = i\omega B_z = 0$, so $\exists \Phi \ni \vec{E}_t = -\vec{\nabla}_t \Phi$ (though Φ might not be single valued) and $\nabla^2\Phi = 0$.
As $\left.\vec{E}_\parallel\right|_S = 0$, $\Phi =$ constant on each boundary. If cross section simply connected, $\Phi =$ constant, $\vec{E} = 0$
No TEM modes on simply connected cylinder
Yes TEM modes on coaxial cable, or two parallel wires.

TE and TM modes

Equations (5) and (6) simplify for

- TM modes, $B_z = 0$, $\gamma^2 \vec{E}_t = ik\vec{\nabla}_t E_z$, $\gamma^2 \vec{B}_t = i\mu\epsilon\omega\hat{z}\times\vec{\nabla}_t E_z$, so $\vec{H}_t = \epsilon\omega k^{-1}\hat{z}\times\vec{E}_t$.
- TE modes, $E_z = 0$, $\gamma^2 \vec{B}_t = ik\vec{\nabla}_t B_z$, $\gamma^2 \vec{E}_t = -i\omega\hat{z}\times\vec{\nabla}_t B_z$, so

$$\vec{E}_t = -\omega\hat{z}\times\vec{B}_t/k \underset{\hat{z}\times}{\Longrightarrow} H_t = k\hat{z}\times E_t/\mu\omega.$$

In either case, $\vec{H}_t = \frac{1}{Z}\hat{z}\times\vec{E}_t$, with

$$Z = \begin{cases} k/\epsilon\omega = (k/k_0)\sqrt{\mu/\epsilon} & \text{TM} \\ \mu\omega/k = (k_0/k)\sqrt{\mu/\epsilon} & \text{TE} \end{cases}$$

To Summarize

Solutions given by $\psi(x,y)$, with $\left(\nabla_t^2+\gamma^2\right)\psi=0$, $\gamma^2=\mu\epsilon\omega^2-k^2$, by

$$\begin{array}{lll}
\text{TM:} & E_z=\psi e^{ikz-i\omega t}, & \vec{E}_t=ik\gamma^{-2}\vec{\nabla}_t\psi e^{ikz-i\omega t} \\
 & \text{with} \quad \psi|_\Gamma=0 & \\
\text{TE:} & H_z=\psi e^{ikz-i\omega t}, & \vec{H}_t=ik\gamma^{-2}\vec{\nabla}_t\psi e^{ikz-i\omega t} \\
 & \text{with} \quad \hat{n}\cdot\vec{\nabla}_t\psi|_\Gamma=0 &
\end{array}$$

By looking at $0=\int_A\psi^*\left(\nabla_t^2+\gamma^2\right)\psi$ we can show $\gamma^2\geq 0$. There are solutions for **discrete** values γ_λ, so only certain wave numbers k_λ for a given frequency can propagate:

$$k_\lambda^2=\mu\epsilon\omega^2-\gamma_\lambda^2,$$

and only frequencies $\omega>\omega_\lambda:=\gamma_\lambda/\sqrt{\mu\epsilon}$ can propagate, and $k_\lambda<\sqrt{\mu\epsilon}\,\omega$, the infinite medium wavenumber. Phase velocity $v_p=\omega/k_\lambda$ is greater than in the infinite medium.

Example: Circular Wave Guide

Jackson does rectangle. You should too. Needed to do homework.

We will consider a circular pipe of (inner) radius r. Of course we should use polar coordinates ρ, ϕ, with

$$\nabla_t^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}, \qquad \text{try} \quad \psi(\rho,\phi) = R(\rho)\Phi(\phi),$$

$$\left(\nabla_t^2 + \gamma^2\right)\psi = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2 R(\rho)\right)\Phi(\phi) + \frac{1}{\rho^2}R(\rho)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

Divide by $R(\rho)\Phi(\phi)$ and multiply by ρ^2:

$$\frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) + \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

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$$\left(\nabla_t^2 + \gamma^2\right)\psi = \left(\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2 R(\rho)\right)\Phi(\phi) + \frac{1}{\rho^2}R(\rho)\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = 0.$$

Divide by $R(\rho)\Phi(\phi)$ and multiply by ρ^2:

$$\begin{aligned} \frac{1}{R(\rho)}\left(\rho\frac{\partial}{\partial\rho}\rho\frac{\partial R}{\partial\rho} + \gamma^2\rho^2 R(\rho)\right) &= C \\ \frac{1}{\Phi(\phi)}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} &= -C. \end{aligned}$$

Solving it

Φ first:

$$\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + C\Phi(\phi) = 0$$

$\Phi(\phi) = e^{\pm i\sqrt{C}\phi}$. Periodicity $\Longrightarrow \sqrt{C} = m \in \mathbb{Z}$.

Now $R(\rho)$:

$$\left(\rho \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} + \gamma^2 \rho^2 - m^2\right) R(\rho) = 0$$

Bessel equation, solutions regular at origin are

$$R(\rho) \propto J_m(\gamma \rho), \quad \text{so} \quad \psi(\rho, \phi) = \sum_{m,n} A_{m,n} J_m(\gamma_{mn} \rho) e^{im\phi}.$$

γ_{mn} is determined by boundary conditions...

Boundary conditions:

For TM, $\psi(r,\phi) = 0 \Longrightarrow J_m(\gamma r) = 0$, so $\gamma_{mn}^{\rm TM} = x_{mn}/r$ where x_{mn} is the n'th value of $x > 0$ for which $J_m(x) = 0$, given on page 114.

For TE, $\hat{n} \cdot \vec{\nabla}_t \psi(r,\phi) = 0 \Longrightarrow \frac{dJ_m}{dr}(\gamma r) = 0$, so $\gamma_{mn}^{\rm TM} = x'_{mn}/r$ where x'_{mn} is the n'th value of $x > 0$ for which $dJ_m(x)/dx = 0$, given on page 370.

Thus the lowest cutoff frequency is the $m = 1$ TE mode, with $x'_{11} = 1.841$ while the lowest TM mode or circularly symmetric mode has $x_{01} = 2.405$.

For a waveguide 5 cm in diameter, with air or vacuum inside, the cutoff frequencies are $f = \frac{\omega}{2\pi} = 3.5$ GHz for the lowest TE and 4.6 GHz for the lowest TM modes.