Ph 441/541   Problem Set 6

Due: Friday, March 30, 2012

1. Convection:

   a. In the mixing-length theory of convection, the mixing length, ℓ, is usually taken to be a constant of order one times the pressure scale height:

   $$h \equiv \left| \frac{1}{P} \left(\frac{dP}{dr} \right) \right|^{-1} .$$

   Use hydrostatic equilibrium to derive an expression for the scale height that involves g, T, the mean molecular weight, and the radius, r. Assume an ideal gas.

   b. The Sun has a convection zone that extends from $r = (0.713 \pm 0.001)\mathcal{R}_\odot$ to just below the photosphere. The table below (after part e) gives the properties of the Sun at a radius near the base of this convection zone. The values were taken from http://www.ap.stmarys.ca/~guenther/evolution/ssm1998.html. Calculate $h/\mathcal{R}_\odot$ for the Sun at the tabulated radius.

   c. Calculate the radiative and adiabatic temperature gradients at the position in the table, assuming an ideal gas. Would you conclude that the Sun is convective at this radius on the basis of these gradients?

   d. Assume that the entire luminosity of the sun is carried by convection at the tabulated radius and calculate the numerical value of $\Delta(\nabla T) \equiv (dT/dr)_{ad} - (dT/dr)_{rad})$ required. Use the mixing-length expression for the convective flux given in part d) of problem 3, assuming that the mixing length, ℓ, is equal to the scale height. Note the the specific heat per mass in that formula is equal to $5N_A k/(2\mu)$ for an ideal gas, where N_A is Avogadro's constant and μ is the mean mass per particle in units of the atomic mass unit — this expression gives the specific heat per gram, you then need to convert to per kilogram. What is the ratio of $\Delta(\nabla T)$ to $(dT/dr)_{ad}$?

   e. (Ph 541 students only) Find the average velocity of the convective bubbles using the results from problem 3. How does it compare with the sound speed?

$r/\mathcal{R}_\odot$	$m(r)/\mathcal{M}_\odot$	$L(r)/\mathcal{L}_\odot$	T(r)	$\rho(r)$	κ	μ
0.722	0.9775	1.0000	2.094×10^6	176 kg m^{-3}	2.24 m^2 kg^{-1}	0.6067

2. Alpha Decay:

We have seen that the quantum mechanical penetration of a Coulomb barrier plays a crucial role in thermonuclear fusion. It also plays a crucial role in the alpha decay of nuclei such as ^{235}U. In the simplest model for alpha decay, the alpha particle is pre-formed and trapped within the nucleus by a potential similar to that shown in Phillips Fig. 4.1. The mean rate of decay, λ, is then the frequency, ν, with which the alpha particle hits the confining barrier times the probability of penetration of the Coulomb barrier; this probability is given by Phillips Eq. (4.12). Write down an approximate expression for the decay rate in terms of ν, E_G, and the energy E released by the alpha decay. The half-life for the alpha decay of ^{235}U is $\tau_{1/2} = 0.69/\lambda = 7.2 \times 10^8$ years and the energy released is $E = 4.68$ MeV. The energy released by the alpha decay of ^{239}Pu is 5.24 MeV. Estimate the half-life of this isotope of plutonium.

3. Convective energy flux: (Ph 541 students only):
Since rising "bubbles" of convecting gas become hotter than their surroundings and sinking bubbles become cooler, energy is transported outwards by convection. This problem examines the "mixing-length" theory for this energy flux. In this model, bubbles of gas are assumed to move a length ℓ (the mixing length) before dissolving into their surroundings.

a. Consider a cube of side s which moves upwards at speed v by a distance equal to the length of its sides. At its upper position it is ΔT warmer than its surroundings. Express the energy transported upwards in terms of the specific heat per unit mass, c_P, the density of the gas, ρ, the volume of the cube, and ΔT. Describe why the specific heat in this answer is the specific heat at constant pressure, c_P. Then express the flux of energy (energy/area/time) carried by the cube in terms of c_P, ρ, ΔT, and the velocity of the cube, v.

b. Express ΔT in terms of the difference between the adiabatic and true temperature gradients (this difference is usually written as $\Delta(\nabla T)$) and the displacement of the cube, s. Assume that the bubble is adiabatic.

c. Use mechanical energy conservation to argue that the velocity of the cube after a displacement by a distance s is given by

$$\frac{1}{2}\rho v^2 = -\frac{1}{2} g(\Delta\rho)s, \tag{1}$$

where g is the gravitational acceleration and $\Delta\rho$ is the difference in density between the cube and its surroundings. Assume that $\Delta\rho$ increases linearly with displacement. Assume that the typical bubble passing a point in the star has moved half of its total path ℓ and show that the typical velocity is

$$v = \left(-\frac{1}{2} g\ell\frac{\Delta\rho}{\rho}\right)^{1/2}. \tag{2}$$

d. Assume an ideal gas with a uniform composition. Put all of the pieces from above parts together to show that the convective energy flux is

$$\mathcal{F}_c = c_P \rho T (g)^{1/2} \left(\frac{\Delta(\nabla T)}{T}\right)^{3/2} \left(\frac{\ell}{2}\right)^2. \tag{3}$$