Physics 601

HW # 2 - Solutions

1. Velocities in the Free Electron Theory. (Simon 4.2)

a) 3d
$$N = \frac{2 \, \frac{4}{3} \pi k_F^3}{\frac{(2\pi)^3}{V}}$$

$$\frac{N}{V} = \frac{k_F^3}{3\pi^2} \Rightarrow k_F = (3\pi^2 n)^{1/3}$$

$$\boxed{v_F = \frac{\hbar k_F}{m} = \frac{\hbar}{m}(3\pi^2 n)^{1/3}}$$

b) $j = \sigma E$ $\qquad \sigma = \frac{ne^2 \tau}{m}$

$$n e v_d = \sigma E$$

$$\boxed{v_d = \frac{\sigma E}{ne}}$$

$$\tau = \frac{\lambda}{v_F} \Rightarrow \boxed{\sigma = \frac{ne^2 \lambda}{m v_F}}$$

Cu at 300 K $\quad E = 1V/m$.

$n = 8.45 \times 10^{28}\ m^{-3}$.

$\sigma = 5.9 \times 10^{7}\ \Omega^{-1}\ m^{-1}$ at 300 K

i) $v_d = \dfrac{(5.9 \times 10^{7})(1)}{(8.45 \times 10^{28})(1.6 \times 10^{-19}\ C)} = \dfrac{(5.9)}{(8.45)(1.6)} \cdot 10^{-2}\ m/s$

$$\boxed{v_d = 4.36 \times 10^{-3}\ m/sec.}$$

$v_F = \dfrac{\hbar}{m}(3\pi^2 n)^{1/3}$.

$= \dfrac{1.05 \times 10^{-34}}{9 \times 10^{-31}}(3\pi^2 \cdot 8.45 \times 10^{28})^{1/3}$.

$= \dfrac{1.05}{9} \cdot 10^{-3}(3\pi^2 \cdot 8.45 \cdot 10^{1})\ 10^{9}$

$= \dfrac{1.05}{9} \cdot 10^{6}\{2502\}^{1/3}$.

$c = 3 \times 10^{8}\ m/s$

$\boxed{v_F \sim 1.58 \times 10^{6}\ m/s} \sim \dfrac{1}{200}\ c$

$$\frac{v_d}{v_F} = \frac{4.36 \times 10^{-3} \ m/s}{1.58 \times 10^{6} \ m/s} = 2.8 \times 10^{9} \ !$$

We note that v_d, the drift velocity, refers to the shift $\left(\frac{m v_d}{\hbar}\right)$ of the center of mass of the entire Fermi sphere and is indeed very small compared to the velocity of the most energetic electrons.

ii) $$\lambda = \frac{m v_F}{n e^2} \sigma$$

$$= \frac{(9 \times 10^{-31})(1.58 \times 10^{6})}{(8.45 \times 10^{28})(1.6 \times 10^{-19})^2} \quad (5.9 \times 10^{7})$$

$$= \frac{(9)(1.58)(5.9)}{(8.45)(1.6)^2} \quad \frac{10^{-18}}{10^{-10}}$$

$$\lambda = 3.9 \times 10^{-8} \ m$$

Mean atom spacing in Cu $\sim 1 \ \mathring{A} = 10^{-10} \ m$

$\lambda \sim 400$ atomic spacings !

2. (Simon # 4.4.)

Another Review of the Free Electron Theory.

• Free Electron Model of a Metal

Here I assume the free electron model refers to the Sommerfeld theory.

A metal is viewed as a gas of noninteracting fermions with spin 1/2 in an infinite square well potential.

• Fermi energy E_F = chemical potential at $T=0$

$$E_F = \frac{\hbar^2 k_F^2}{2m}$$

Fermi temperature $T_F = \frac{E_F}{k_B}$.

• Even at temperatures $0 < T \ll T_F$, some electrons are thermally excited. These electrons transport both electricity and heat as evidenced by the Wiedemann-Franz ratio.

(a) <u>d-dimensional sample</u> w/ volume L^d.

<u>w/ N electrons.</u>

$$N = \underset{\text{spin}}{\underset{\downarrow}{2}} \frac{c_d\, k_F^d}{\left(\frac{2\pi}{L}\right)^d}$$

$$N = 2 \left(\frac{L}{2\pi}\right)^d c_d\, k_F^d$$

$$\Downarrow$$

$$k_F = \left(\frac{N}{2c_d}\right)^{1/d} \frac{2\pi}{L}$$

$$E_F = \frac{\hbar^2}{2m} \left(\frac{2\pi}{L}\right)^2 \left(\frac{N}{2c_d}\right)^{2/d}$$

$$= \frac{\hbar^2}{2mL^2} \left(\frac{N\,(2\pi)^d}{2\,c_d}\right)^{2/d}$$

$$= \frac{\hbar^2}{2mL^2} \left(N a_d\right)^{2/d} \qquad a_d = \frac{(2\pi)^d}{2c_d}$$

$d=1.$ $\quad c_d = 2$

$$a_d = \frac{\pi}{2}.$$

$d=2$ $\quad c_d = \pi.$

$$a_d = \frac{(2\pi)^2}{2\pi} = 2\pi$$

$d=3$ $\quad c_d = \frac{4}{3}\pi$

$$a_d = \frac{(2\pi)^3}{2\,\frac{4}{3}\pi} = 3\pi^2$$

(v). $g(\varepsilon_F)$

We've just found that

$$E_F = \frac{\hbar^2}{2mL^2}(N a_d)^{2/d}$$

density of electrons N/V

$$= \frac{\hbar^2}{2m}\left(\frac{N}{L^d}a_d\right)^{2/d} = \frac{\hbar^2}{2m}\left(n\; a_d\right)^{2/d}$$

$$\ln \varepsilon_F = \frac{2}{d} \ln n + \text{const}$$

We differentiate each side to obtain

$$\frac{d\varepsilon_F}{\varepsilon_F} = \frac{2}{d} \frac{dn}{n}$$

$$\Downarrow$$

$$\frac{dn}{d\varepsilon_F} = g(\varepsilon_F) = \frac{d}{2} \frac{n}{E_F} = \frac{d}{2L^d} \frac{N}{\varepsilon_F}$$

$$\boxed{g(E_F) = \frac{Nd}{2L^d E_F}}$$

1-d $\quad n = N/L = 1/.8 \text{ nm} = 1.25 \times 10^{+9}$

$$E_F = \frac{\hbar^2}{2mL^2} (Na_1)^2$$

$$= \frac{\hbar^2}{2m} n^2 \left(\frac{\pi}{2}\right)^2$$

$$E_F = \frac{(1.05 \times 10^{-34})^2}{2(9 \times 10^{-31})} (1.25 \times 10^{9})^2 \left(\frac{\pi}{2}\right)^2$$

$$= \frac{(1.05)^2 (1.25)^2 \pi^2}{(2)(9)(2)^2} \cdot \frac{10^{-68}\ 10^{18}}{10^{-31}}$$

$$= .236 \times 10^{-19} \text{ J}$$

$$= 2.36 \times 10^{-20} \text{ J} \quad \frac{1\text{ eV}}{1.6 \times 10^{-19} \text{ J}} = .14 \text{ eV}$$

$$\boxed{E_F = .14 \text{ eV}}$$

$$T_F = \frac{E_F}{k_B}$$

$$= \frac{2.4 \times 10^{-20} \text{ J}}{1.4 \times 10^{-23} \text{ J/K}} = 1.7 \times 10^{3} \text{ K}$$

$$\boxed{T_F = 1.7 \times 10^{3} \text{ K}}$$

(c) $E = c|p|$

From (a) we have

$$k_F = \left(\frac{N}{2c_d}\right)^{1/d} \frac{2\pi}{L}$$

Now $E_F = c(\hbar k_F) = \frac{\hbar c}{\cancel{m}} \frac{2\pi}{L} \left(\frac{N}{2c_d}\right)^{1/d}$

$$E_F = 2\pi\hbar c \left(\frac{N}{L^d 2c_d}\right)^{1/d} \qquad n = \frac{N}{L^d}$$

$$\boxed{E_F = \frac{2\pi\hbar c}{1}\left(\frac{n}{2c_d}\right)^{1/d}}$$

$$\ln E_F = \frac{1}{d}\ln n + \text{constant}$$

$$\frac{dE_F}{dE_F} = \frac{1}{d}\frac{dn}{n} \Rightarrow \frac{dn}{dE_F} = g(\varepsilon_F) = \frac{nd}{\varepsilon_F}$$

$$\boxed{g(E_F) = \frac{nd}{E_F}}$$

$d = 1 \qquad C_d = 2.$

$$E_F = \frac{2\pi\hbar c}{1}\left(\frac{n}{2C_d}\right)^{1/d} = \frac{2\pi\hbar c}{1}\left(\frac{n}{4}\right)^{1}$$

$$\boxed{E_F = \frac{\pi\hbar c\, n}{2}} \qquad d = 1$$

$$g(E_F) = \frac{nd}{\varepsilon_F} \Rightarrow \boxed{g(E_F) = \frac{n}{\varepsilon_F}} \qquad d = 1$$

$d = 2 \qquad C_d = 4\pi$

$$E_F = \frac{2\pi\hbar c}{1}\left(\frac{n}{2C_d}\right)^{1/d} = \frac{2\pi\hbar c}{1}\left(\frac{n}{8\pi}\right)^{1/2}$$

$$\boxed{E_F = \frac{\hbar c}{1}\left(\frac{\pi n}{2}\right)^{1/2}} \qquad d = 2.$$

$$g(E_F) = \frac{nd}{\varepsilon_F} \Rightarrow \boxed{g(E_F) = \frac{2n}{\varepsilon_F}} \qquad d=2.$$

$$d=3 \qquad c_d = \frac{4}{3}\pi$$

$$E_F = \frac{2\pi\hbar c}{1}\left(\frac{n}{2c_d}\right)^{1/d} = \frac{2\pi\hbar c}{m}\left(\frac{3n}{8\pi}\right)^{1/3}.$$

$$\boxed{E_F = \left(\frac{2\pi\hbar c}{1}\right)\left(\frac{3n}{8\pi}\right)^{1/3}} \qquad d=3$$

$$g(\varepsilon_F) = \frac{nd}{\varepsilon_F} \Rightarrow \boxed{g(E_F) = \frac{3n}{\varepsilon_F}} \qquad d=3.$$

3. (Simon Problem 4.5)

Chemical Potential of 2D Electrons.

$$N = \int_0^\infty \frac{d\varepsilon \quad g(\varepsilon)}{e^{(\varepsilon-\mu)/k_B T} + 1}$$

$$g(\varepsilon) = 2\frac{dN}{d\varepsilon} = 2\frac{dN}{dk}\frac{dk}{dE}$$

2d $\quad N(k) = \frac{\pi k^2}{\left(\frac{2\pi}{L}\right)^2} = \frac{A\pi k^2}{4\pi^2} = \frac{Ak^2}{4\pi}$

$$\frac{dN}{dk} = \frac{Ak}{2\pi}$$

$$E = \frac{\hbar^2 k^2}{2m} \Rightarrow \frac{dE}{dk} = \frac{\hbar^2 k}{m}$$

$$g(\varepsilon) = \cancel{2}\frac{A\cancel{k}}{\cancel{2}\pi}\frac{m}{\hbar^2\cancel{k}} = \frac{Am}{\pi\hbar^2}.$$

Then in two dimensions we have

$$n = \frac{m}{\pi\hbar^2}\int_0^\infty \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1} = \frac{m}{\pi\hbar^2}\left(-\frac{1}{\beta}\right)\int_0^\infty \frac{e^{-\beta(\varepsilon-\mu)}(-\beta)\,d\varepsilon}{1+e^{-\beta(\varepsilon-\mu)}}$$

$$\Downarrow$$

$$\beta \equiv \frac{1}{k_B T} \qquad n = -\frac{m}{\pi\hbar^2\beta}\left\{\ln\left(1+e^{-\beta(\varepsilon-\mu)}\right\}\right]_0^\infty$$

$$n = -\frac{m}{\pi\hbar^2\beta}\left\{0 - \ln\left(1+e^{\beta\mu}\right)\right\}$$

$$\frac{\pi\hbar^2\beta n}{m} = \ln\left\{1+e^{\beta\mu}\right\} \Rightarrow \beta\mu = \ln\left\{e^{\frac{\pi\hbar^2\beta n}{m}} - 1\right\}$$

$$T \ll \mu \Rightarrow e^{\frac{\pi\hbar^2\beta n}{m}} \gg 1$$

(T low)

$$\Downarrow$$

$$\frac{\mu}{\cancel{kT}} \Rightarrow \frac{\pi\hbar^2 n}{m\cancel{kT}} \Rightarrow \boxed{\mu = \frac{n\hbar^2\pi}{m}}$$

Independent of T

$T \ll \mu$ (T low).

4. Madelung's Rule (Simon 5.1).

a) Atomic filling configuration for W $(Z = 74)$ - Tungsten

$\# e's = 2(2l+1)$.

n	s	p	d	f	g	
1	2					2
2	2	6				4
3	2	6	10			12
4	2	6	10	14		20.
5	2	6	10	14	16	38.
6	2		↑			56
			take $4e^-$s			70

$\tilde{n} \equiv n + l$

$$\boxed{\begin{aligned} [W] &= \underbrace{1s^2}_{1}\ \underbrace{2s^2}_{2}\ \underbrace{2p^6\,3s^2}_{3}\ \underbrace{3p^6\,4s^2}_{4}\ \underbrace{3d^{10}\,4p^6\,5s^2}_{5} \\ &\quad \underbrace{4d^{10}\,5p^6\,6s^2}_{6}\ 4f^{14}\ 5d^4 \end{aligned}}$$

($\tilde{n}$: 1, 2, 3, 4, 5, 6)

h) Element 118 is a noble gas → what is the atomic # of the next one.

n	s	p	d	f	g		
4	2	6	10	14 ←	70		
5	2	6	10	14	18		
6	2	6	10	14	18	22	
7	2	$\overset{118}{6}$	10	14	18		88 118
8	2	6					

Element 118 full up to $7p^6$

Next noble gas full up to $8p^6$

We must do the addition

$$118 + 2 + 18 + 14 + 10 + 6 = 118 + 50 = 168$$

Next noble gas $Z = 168$

5. Chemical Bonding (Simon 6.1).

(a) Five different types of bonding and atoms where expected to occur.

(i) Ionic

Electron transfer from one atom to another => ions have electrostatic interaction

Occurs between elements w/ different electronegativities (.eg I – VII compounds like NaCl)

(ii) Covalent

Sharing of electron by the two atoms in the bond

Occurs between elements of very similar electronegativities or in solids of one element
(two examples: diamond (C) or GaAs – III – V compound)

(iii) Metallic

Delocalization of valence electrons throughout solid

L and middle of periodic table.

(iv) Van der Waals.

Dipolar interaction w/ no electron transfer.

Noble gas solids, Solids composed of nonpolar molecules.

(v) H bond

H is bound to one atom but attracted to another - quite weak but long-ranged.

Very important in biological molecules. (also in ice and more generally in different phases of H_2O).

(b) Van der Waals forces.

The van der Waals force between two atoms (or molecules) results from the interaction of their dipole moments, either permanent or fluctuating. If one atom has a dipole moment $\vec{p}_1$ in the $\hat{z}$ direction, a second one will sense an electric field

$$\vec{E} = -\frac{p_1}{4\pi\varepsilon_0 r^3}\hat{z}$$

and will then develop a dipole moment $\vec{p}_2 = \chi\vec{E}$.

The potential energy of these two dipoles is then

$$U \propto -\frac{|p_1||p_2|}{r^3} \propto -\frac{p_1^2 \chi E}{r^3} \propto -\frac{p_1^2 \chi}{r^6}$$

Therefore $\boxed{F = -\frac{dU}{dr} \propto \frac{1}{r^7}}$ and is attractive.

6.

a)

$$\Delta KE = \underbrace{\frac{m}{2}\left(\vec{v} - \frac{e\vec{E}t}{m}\right)^2}_{\substack{\text{kinetic energy}\\ \text{before second}\\ \text{collision}}} - \underbrace{\frac{m}{2}(\vec{v}\,')^2}_{\substack{\text{kinetic energy}\\ \text{just after}\\ \text{second collision}}}$$

$$= \underbrace{\frac{m}{2}(v^2 - v'^2)}_{\substack{= 0\\ \text{(assumption of}\\ \text{Drude model)}}} + \frac{m}{2}\left(\frac{eEt}{m}\right)^2 - \underbrace{\frac{met}{m}\,\vec{E}\cdot\vec{v}}_{\substack{\vec{v}\text{ averaged over}\\ \text{spherically}\\ \text{symmetric}\\ \text{distribution} \Rightarrow 0}}$$

$\therefore$ Average energy loss after 2 collisions $= \dfrac{(eEt)^2}{2m} = \Delta E$

b) τ = mean time between two collisions for a single electron

$$\frac{\text{Average energy loss to ions}}{\text{electron}\cdot\text{collision}} = \underset{\substack{\downarrow \\ \#\text{ collisions per} \\ \text{"event" in a)}}}{2} \cdot \underset{\substack{\downarrow \\ \text{average energy} \\ \text{loss/e after} \\ \text{2 collisions} \\ \text{(from a)}}}{\frac{(eE\tau)^2}{2m}}$$

More formally

$$\langle \Delta E \rangle = \int_0^\infty \frac{dt}{\tau} e^{-t/\tau} \Delta E = \frac{(eE\tau)^2}{m}$$

(since Prob. of time between collisions is $\frac{dt}{\tau} e^{-t/\tau}$)

$$\boxed{= \frac{(eE\tau)^2}{m}}$$

$$\frac{\text{Average energy loss}}{\text{cm}^3 - \text{sec}} = \underset{\substack{\downarrow \\ \#\text{ collisions/sec-cm}^3}}{\frac{n}{\tau}} \quad \underset{\substack{\downarrow \\ \text{energy loss/e-collision}}}{\frac{(eE\tau)^2}{m}}$$

$$\frac{\underline{\text{Average energy loss}}}{\text{cm}^3 - \text{sec}} = \underbrace{\frac{ne^2\tau}{m}}_{\sigma} E^2$$

$\downarrow$

$$\frac{[P]}{[L^3]} \quad \text{since } [P] = \frac{[E]}{[T]}$$

$$= (\sigma E)\, E = \frac{j^2}{\sigma}$$

$$\frac{P}{LA} = \frac{(I/A)^2 \, RA}{L} \Rightarrow \boxed{P = I^2 R}$$

where we have used $j = I/A$ Joule Heating!

$$\sigma = 1/\rho$$

$$\rho = RA/L$$

Challenges associated w/ observing quantum degeneracy in Fermi gases of cold atoms and how this was achieved (from Jin, "A Fermi Gas of Atoms").

- Cooling and equilibration of gases at low temperature achieved through hands-on "s-wave" collisions - not possible for fermions, due to Pauli exclusion principle, which makes it challenging to achieve low temperatures in equilibrium
- Successful cooling of gases of fermionic atoms using s-wave collisions of mixtures of atoms that are in different states
- Jin's group used atoms in two distinct spin states
- Hulet's group used mixture of isotopes.
- Evaporative cooling used for these mixtures to achieve quantum degeneracy via collisions.

Fermi gases in astrophysics.

a) N_e = # electrons

$$N_p \sim \frac{M_\odot}{m_p} = \frac{2 \times 10^{33}\ g}{1.7 \times 10^{-24}\ g} \sim 10^{57}.$$

~~# protons~~ in sun

Let us assume that there are roughly the same number of e's and p's

$\Downarrow$

$$\mathcal{E}_F = \frac{\hbar}{2m} \left(3\pi^2 n\right)^{2/3}$$

where

$$n = \frac{N_e}{V} \quad \text{where} \quad V = \frac{4}{3}\pi R_s^3$$

$$= \frac{4}{3}\pi \left(2 \times 10^9\right)^3.$$

$$\sim 3 \times 10^{28}\ cm^3.$$

$$n \sim \frac{10^{57}}{3 \times 10^{28}} \sim 3 \times 10^{28}\ \text{electrons}/cm^3.$$

$$\Downarrow$$

$$\varepsilon_F \sim \frac{(10^{-27})^2}{2(9\times 10^{-28})} \left\{ 3\pi^2 (3\times 10^{-28}) \right\}^{2/3}$$

$$\sim \frac{1}{2}(10^{-27})(10^{20}) \sim 5\times 10^{-6} \text{ ergs}.$$

$$\varepsilon_F \sim 5\times 10^{-6} \text{ ergs} \times \frac{1 \text{ eV}}{1.6\times 10^{-12} \text{ ergs}}$$

$$\Downarrow$$

$$\boxed{\varepsilon_F = 3.6 \times 10^{4} \text{ eV}}$$

b) k_F is not affected by relativity

In 3d we determine k_F

$$\Downarrow$$

$$N = \frac{2\,\frac{4}{3}\pi k_F^3}{(2\pi)^3/V} \Rightarrow k_F \sim \left(\frac{N}{V}\right)^{1/3}.$$

Then in the relativistic limit

$\Downarrow$

$$\boxed{\varepsilon_F = \hbar k_F c \sim \hbar c \left(\frac{N}{V}\right)^{1/3}.}$$

c) Now $\tilde{R}_s = 10 \text{ km} = 10^6 \text{ cm}$.

$(R_s = 2 \times 10^9 \text{ cm})$

$$n \sim 3 \times 10^{28} \frac{e}{\text{cm}^3} \times \frac{(2 \times 10^9)^3}{10^{18}}$$

$$\sim 2.4 \times 10^{38} \; e/\text{cm}^3$$

$$\varepsilon_F \sim \hbar c n^{1/3} \sim (10^{-27})(3 \times 10^{10})(10^{13})$$

$$\sim 2 \times 10^{-4} \text{ erg} \cdot \frac{1 \text{ eV}}{1.6 \times 10^{-12} \text{ erg}}$$

$\Downarrow$

$\boxed{\varepsilon_F \sim 10^8 \text{ eV}}$ relativistic

$(m_e c^2 \sim .51 \times 10^6 \text{ eV})$.

Liquid He^3.

$$\varepsilon_F = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}.$$

$$\rho = .081 \ g/cm^3$$

$$\frac{\#\ moles}{cm^3} = \frac{1}{3}\ 81 \times 10^{-3}$$

$$= 27 \times 10^{-3} = 2.7 \times 10^{-2} \ \frac{moles}{cm^3}.$$

$$n_H = \text{concentration of atoms} = 2.7 \times 10^{-2} \ \frac{moles}{cm^3}$$

$$\times \ 6 \times 10^{23} \ atoms/mole$$

$$= 1.6 \times 10^{22} \ atoms/cm^3.$$

$$m_H = (3)\ m_p = (3)(1.6 \times 10^{-24} \ g)$$

$$\sim 5 \times 10^{-24} \ g.$$

$$\mathcal{E}_F \sim \frac{(10^{-27})^2}{2.5 \times 10^{-24}} \left\{ 3 \cdot \pi^2 \, (1.6 \times 10^{22}) \right\}^{2/3}$$

$$\sim \underbrace{\frac{10^{-54}}{10^{-23}}}_{} \left\{ (3\pi^2)(16)(10^{21}) \right\}^{2/3}$$

$$\sim 10^{-31} \left[[30][16] \right]^{2/3} \cdot 10^{14}$$

$$\boxed{\mathcal{E}_F \sim 6 \times 10^{-16} \text{ erg}}$$

$$\boxed{T_F = \frac{\mathcal{E}_F}{k_B} \sim \frac{6 \times 10^{-16} \text{ erg}}{1.4 \times 10^{-16} \text{ erg/K}} \sim 4.29 \text{ K}}$$

Cohesive Energy of Free Electron Fermi Gas.

10. a) $T=0$ Average $KE/e = \frac{3}{5} E_F$

$$= \frac{\frac{3}{5}\frac{\hbar^2}{2m}(3\pi^2 n)}{\frac{me^4}{2\hbar^2}} \qquad 1\ Ry$$

$$= \frac{3}{5}\frac{\hbar^4}{m^2e^4}(3\pi^2 n)^{2/3}$$

$$= \frac{3}{5} a_H^2 \left(\frac{3\pi^2 \cdot 3}{4\pi r_0^3}\right)^{2/3} \qquad a_H = \frac{\hbar^2}{me^2}$$

$$= \left(\frac{3}{5}\right)\left(\frac{9\pi}{4}\right)^{2/3}\frac{1}{r_s^2} \qquad \frac{4\pi r_0^3}{3} = \frac{1}{n}$$

$$\langle KE\rangle/e = \frac{2.21}{r_s^2} \qquad r_s \equiv \frac{r_0}{a_H}$$

b) Coulomb energy

$$U_c^{in} = e \int_0^{r_0} \frac{\rho}{r} \, 4\pi r^2 \, dr = -\frac{3e^2}{2r_0}$$

$$\left(\frac{4\pi r_0^3}{3}\right) = -\frac{e}{\rho}$$

$$U_c^{in} = \frac{-\dfrac{3\cancel{e^2}}{\cancel{2}r_0}}{\dfrac{\cancel{e^2}}{\cancel{2}a_H}} \quad 1 \text{ Ry}$$

$$U_c^{in} = -\frac{3}{r_s} \text{ Rydberg.}$$

c) Coulomb self-energy

$$U_c^{self} = \rho^2 \int_0^{r_0} dr \left(\frac{4\pi r^3}{3}\right) \frac{(4\pi r^2)}{r} = \frac{3e^2}{5r_0}.$$

$$U_c^{self} = \frac{\frac{3e^2}{5r_0}}{\frac{e^2}{2a_H}} = \frac{6}{5r_s} \text{ Rydbergs.}$$

d) $$U_c^{ion} + U_c^{self} = \frac{-1.80}{r_s}.$$

Sum of Coulomb and kinetic energies is

$$U_{binding} = \frac{-1.80}{r_s} + \frac{2.21}{r_s^2}$$

Minimum when

$$\frac{1.80}{r_s^2} = \frac{4.42}{r_s^3} \Rightarrow r_s = \frac{4.42}{1.8} = 2.45$$

$U_{binding}\,(r_s = 2.45) < 1$ Rydberg
($\sim .4$ Rydberg)

$\Downarrow$

Separated H atoms are more stable.