Midterm 2010 solution

Note Title 10/19/2010

1. B. False

2. B.

3. D. : $H = \epsilon \begin{pmatrix} 0 & 1 \\ -1 & b \end{pmatrix}$, H is not hermitian.
So it cannot be a valid Hamiltonian

4.
(a) $\Psi(x,0) = A(2\psi_0(x) + \psi_1(x))$

$\Rightarrow \langle \Psi | \Psi \rangle = 1 \Rightarrow A^2(4+1) = 1$

$\Rightarrow A = \frac{1}{\sqrt{5}}$

(b) No

(c) $\Psi(x,t) = \frac{1}{\sqrt{5}}\left(2\psi_0(x) e^{-i\frac{E_0}{\hbar}t} + \psi_1(x) e^{-i\frac{E_1}{\hbar}t}\right)$

(d) Yes.

(e) Zero

(f) $\langle H \rangle = \frac{4}{5}E_0 + \frac{1}{5}E_1$

(g) $P(E_1) = \frac{4}{5}$

(h) $\Omega(x,0) = \psi_0(x)$

5. (a) $\langle p \rangle = i\sqrt{\frac{\hbar m\omega}{2}} \langle a_+ - a_- \rangle = 0$

$\langle p^2 \rangle = -\frac{\hbar m\omega}{2} \langle (a_+ - a_-)^2 \rangle$

$= -\frac{\hbar m\omega}{2} \langle a_+^2 - a_+a_- - a_-a_+ + a_-^2 \rangle$

$$= + \frac{\hbar m\omega}{2} \langle a_+ a_- + a_- a_+ \rangle$$

$$= \frac{\hbar m\omega}{2} \langle 2a_+ a_- + 1 \rangle$$

$$= \frac{\hbar m\omega}{2} \langle 2n + 1 \rangle = \hbar m\omega \langle n + \frac{1}{2} \rangle$$

(b)

$$\psi_1(x) = a_+ \psi_0(x)$$

$$= \sqrt{\frac{1}{2\hbar m\omega}} (-ip + m\omega x) A e^{-\frac{m\omega}{2\hbar} x^2}$$

$$= A\sqrt{\frac{1}{2\hbar m\omega}} \left(-\hbar \frac{d}{dx} + m\omega x\right) e^{-\frac{m\omega}{2\hbar} x^2}$$

$$= A\sqrt{\frac{1}{2\hbar m\omega}} \left(\cancel{\hbar} \frac{m\omega}{\cancel{2}\cancel{\hbar}} \cdot \cancel{2} x + m\omega x\right) e^{-\frac{m\omega}{2\hbar} x^2}$$

$$= A\sqrt{\frac{1}{2\hbar m\omega}} \cdot 2m\omega x \; e^{-\frac{m\omega}{2\hbar} x^2}$$

$$= A \sqrt{\frac{2m\omega}{\hbar}} \; e^{-\frac{m\omega}{2\hbar} x^2}$$

6. $$\langle f | i\frac{d}{dx} g \rangle = \int_{-\infty}^{\infty} f^* i \frac{d}{dx} g \, dx$$

$$= i\left[\cancel{f^* g \Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} \frac{df^*}{dx} \cdot g \, dx\right]$$

$$= \int_{-\infty}^{\infty} \left(i\frac{df}{dx}\right)^* g \, dx$$

$$= \langle i\frac{df}{dx} | g \rangle$$

$\therefore$ $i\frac{d}{dx}$ is hermitian

7. (a) $H = \begin{pmatrix} 0 & \epsilon \\ \epsilon & 0 \end{pmatrix}$ with $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(b) $H|\alpha\rangle = \lambda|\alpha\rangle$

$\Rightarrow \det \begin{vmatrix} -\lambda & \epsilon \\ \epsilon & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^2 - \epsilon^2 = 0 \quad , \lambda = \pm\epsilon$

For $\lambda = \epsilon$

$\begin{pmatrix} -\epsilon & \epsilon \\ \epsilon & -\epsilon \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$\Rightarrow -a + b = 0 \quad \Rightarrow b = a$

$\Rightarrow |\alpha_\epsilon\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$

For $\lambda = -\epsilon$

$\begin{pmatrix} \epsilon & \epsilon \\ \epsilon & \epsilon \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$

$\Rightarrow a + b = 0 , \ b = -a$

$\Rightarrow |\alpha_{-\epsilon}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle)$

(c) $|\mathcal{S}(t)\rangle = c_\epsilon e^{-i\frac{\epsilon}{\hbar}t}|\alpha_\epsilon\rangle$

$+ c_{-\epsilon} e^{i\frac{\epsilon}{\hbar}t}|\alpha_{-\epsilon}\rangle$

$= \frac{1}{\sqrt{2}}\begin{pmatrix} c_\epsilon e^{-i\frac{\epsilon}{\hbar}t} + c_{-\epsilon} e^{i\frac{\epsilon}{\hbar}t} \\ c_\epsilon e^{-i\frac{\epsilon}{\hbar}t} - c_{-\epsilon} e^{i\frac{\epsilon}{\hbar}t} \end{pmatrix}$

$$|\psi(0)\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} c_\epsilon + c_{-\epsilon} \\ c_\epsilon - c_{-\epsilon} \end{pmatrix}$$

$$\therefore \quad c_\epsilon = c_{-\epsilon} = \frac{1}{\sqrt{2}}$$

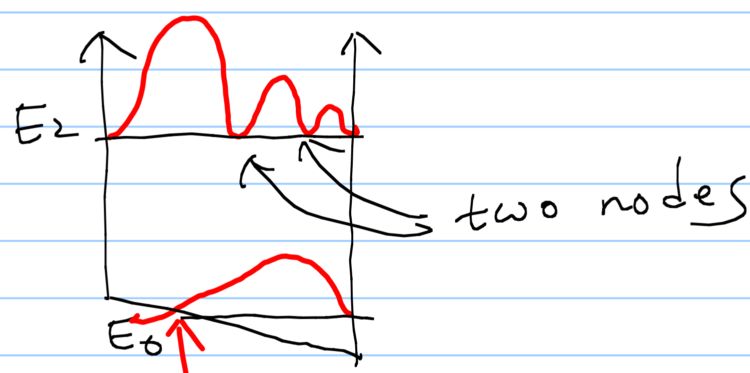
$$\therefore |\psi(t)\rangle = \frac{1}{2} \begin{pmatrix} e^{-i\frac{\epsilon}{\hbar}t} + e^{i\frac{\epsilon}{\hbar}t} \\ e^{-i\frac{\epsilon}{\hbar}t} - e^{i\frac{\epsilon}{\hbar}t} \end{pmatrix}$$

$$= \begin{pmatrix} \cos\left(\frac{\epsilon}{\hbar}t\right) \\ -i\sin\left(\frac{\epsilon}{\hbar}t\right) \end{pmatrix}$$

$$P(|1\rangle) = |\langle 1|\psi(t)\rangle|^2$$

$$= \cos^2\left(\frac{\epsilon}{\hbar}t\right)$$

8.



penetrates

$\longleftarrow$

λ increases

Amplitude increases