Midterm 2009 solution

Note Title 10/18/2009

1.(a) $A = \frac{1}{\sqrt{2}}$

$\Psi(x,0) = \sum_n c_n \psi_n(x)$

$\Rightarrow \sum_n |c_n|^2 = 1$; normalization condition

$\Rightarrow$ Thus $A^2 + A^2 = 1$

$\Rightarrow A^2 = \frac{1}{2} \Rightarrow A = \frac{1}{\sqrt{2}}$

(b) No

$\psi_0(x)$ and $\psi_1(x)$ are each stationary state, but their sum is not.

More clearly,

$\hat{H}\psi_0(x) = E_0 \psi_0(x)$

$\hat{H}\psi_1(x) = E_1 \psi_1(x)$

$\Rightarrow$ However, $\hat{H}(\psi_0 + \psi_1) \neq E(\psi_0 + \psi_1)$

$\hat{H}\Psi(x,0) \neq E\,\Psi(x,0)$

$\Psi(x,0)$ is not an eigen state of the Hamiltonian.

(c) $\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\psi_0(x) e^{-i\frac{E_0}{\hbar}t} + \psi_1(x) e^{-i\frac{E_1}{\hbar}t}\right)$

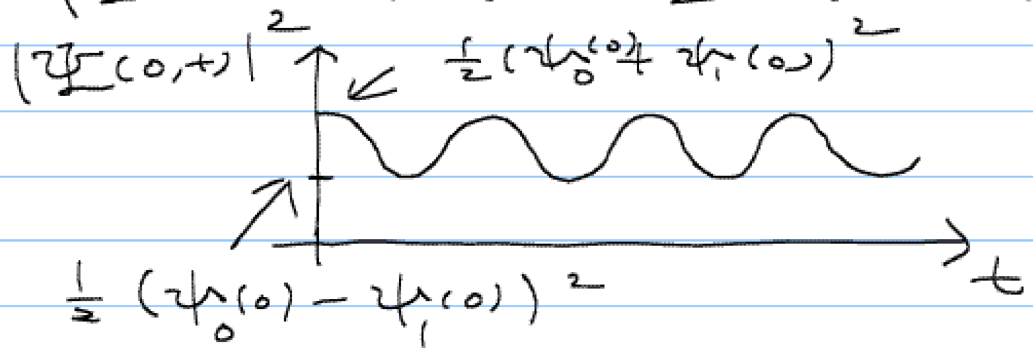
(d) $|\Psi(0,t)|^2 = \frac{1}{2}\left(\psi_0^* e^{i\frac{E_0}{\hbar}t} + \psi_1^* e^{i\frac{E_1}{\hbar}t}\right) \cdot \left(\psi_0 e^{-i\frac{E_0}{\hbar}t} + \psi_1 e^{-i\frac{E_1}{\hbar}t}\right)$

$= \frac{1}{2}\left(\psi_0^2 + \psi_1^2 + \psi_0\psi_1\left(e^{i(E_1-E_0)/\hbar\, t} + e^{-i(E_1-E_0)/\hbar\, t}\right)\right)$

$$= \frac{1}{2}\left(\psi_0^2 + \psi_1^2 + 2\psi_0\psi_1 \cos\left(\frac{E_1 - E_0}{\hbar} t \right) \right)$$

$$|\Psi(0,t)|^2_{max} = \frac{1}{2}(\psi_0(0) + \psi_1(0))^2$$

$$|\Psi(0,t)|^2_{min} = \frac{1}{2}(\psi_0(0) - \psi_1(0))^2$$



(e) Yes : $\Psi(x,t)$ should always satisfy the time dependent Schrödinger Eq. Otherwise, it is not a valid wave-function.

(f) Zero : We can measure only the eigenvalues of the particular observable operator. Therefore, for energies, we can measure only E_0, E_1, ... but never a value other than these.

(g) $\frac{1}{2}(E_0 + E_1)$: When $\Psi(x,t) = \sum_n c_n \psi_n(x) \cdot e^{-i\frac{E_n}{\hbar}t}$

$$\Rightarrow \langle H \rangle = \sum_n |c_n|^2 E_n = \frac{1}{2}E_0 + \frac{1}{2}E_1$$

(i) $\Omega(x,0) = \psi_i(x)$: Measuring E_i collapses the state to its eigenstate $\psi_i(x)$

(j) 6 : Measuring the position "x" collapses the state to the eigenstate of "x", which is a delta function centered at that particular position. Such an eigenstate of "x" can never be expressed as a finite sum of energy eigenstates.

(k) 5 : The position measurement has collapsed the system to the eigenstate of "x". This eigenstate cannot be expressed as any finite sum of the energy eigenstates. Therefore, energy measurement can yield any value of the the energy spectrum.

$\langle n | a_\pm^2 | n \rangle \propto \langle n | n \pm 2 \rangle = 0$

2.(a) $x^2 = \frac{\hbar}{2m\omega}(a_+ + a_-)^2$

$= \frac{\hbar}{2m\omega}(a_+^2 + a_+ a_- + a_- a_+ + a_-^2)$

$\langle x^2 \rangle_n = \frac{\hbar}{2m\omega} \langle n | a_+^2 + a_+ a_- + a_- a_+ + a_-^2 | n \rangle$

$= \frac{\hbar}{2m\omega} \{ \langle n | a_+ a_- + a_- a_+ | n \rangle \}$

$$= \frac{\hbar}{2m\omega}\left\{ \langle n | 2a_+a_- + 1 | n \rangle \right\}$$

$$= \frac{\hbar}{2m\omega}\left\{ 1 + 2n \right\} = \frac{\hbar}{m\omega}\left(n + \frac{1}{2}\right)$$

(b)

$$\psi_0(x) = A e^{-\frac{m\omega}{2\hbar}x^2}$$

$$\psi_1(x) = a_+\psi_0(x) = \frac{1}{\sqrt{2\hbar m\omega}}\left(-ip + m\omega x\right)\psi_0$$

$$= \frac{A}{\sqrt{2\hbar m\omega}}\left(-\hbar\frac{d}{dx} + m\omega x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}$

$$= \frac{A}{\sqrt{2\hbar m\omega}}\left(+\cancel{\hbar}\frac{m\omega}{\cancel{2\hbar}}\cdot\cancel{2}x + m\omega x\right) e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= \frac{A\,2m\omega x}{\sqrt{2\hbar m\omega}} e^{-\frac{m\omega}{2\hbar}x^2}$$

$$= A\sqrt{\frac{2m\omega}{\hbar}}\; x\, e^{-\frac{m\omega}{2\hbar}x^2}$$

3. $$\langle f | \tfrac{d}{dx} g \rangle = \int_{-\infty}^{\infty} f^* \frac{d}{dx} g \, dx$$

$$= \cancel{f^* g \Big|_{-\infty}^{\infty}}^{\,0} - \int_{-\infty}^{\infty} \left(\frac{df}{dx}\right)^* g \, dx$$

$$= \langle -\tfrac{d}{dx} f | g \rangle \neq \langle \tfrac{d}{dx} f | g \rangle$$

$\therefore$ $\frac{d}{dx}$ is not hermitian

4. $$\begin{pmatrix} 1 & 1+i \\ 2 & 2i \end{pmatrix}^\dagger = \left(\begin{pmatrix} 1 & 1+i \\ 2 & 2i \end{pmatrix}^T\right)^* = \begin{pmatrix} 1 & 2 \\ 1-i & -2i \end{pmatrix}$$

5. $$(\hat{x}\hat{p})^\dagger = \hat{p}^\dagger \hat{x}^\dagger = \hat{p}\,\hat{x}$$

6. (a) with $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$H = a|1\rangle\langle 1| + b|2\rangle\langle 2| + c|1\rangle\langle 2| + d|2\rangle\langle 1| = \begin{pmatrix} a & c \\ d & b \end{pmatrix}$$

You can check this by, for example,

$$c|1\rangle\langle 2| = c\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} = c\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$$

Thus $$H = \epsilon\left(|1\rangle\langle 1| + |2\rangle\langle 2| - |1\rangle\langle 2| - |2\rangle\langle 1|\right) = \epsilon\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

(b) $\begin{vmatrix} \epsilon-\lambda & -\epsilon \\ -\epsilon & \epsilon-\lambda \end{vmatrix} = 0$

$\Rightarrow (\epsilon-\lambda)^2 - \epsilon^2 = 0$

$\Rightarrow (\lambda-\epsilon-\epsilon)(\lambda-\epsilon+\epsilon) = 0$

$\Rightarrow \underline{\lambda = 0,\ 2\epsilon}$

For $\underline{\lambda = 0}$ $\quad |\alpha\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} \epsilon & -\epsilon \\ -\epsilon & \epsilon \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow \quad b = a$

$\Rightarrow |\alpha\rangle = \begin{pmatrix} a \\ a \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$|\alpha_0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underline{\frac{1}{\sqrt{2}}\left(|1\rangle + |2\rangle\right)}$

For $\lambda = \underline{2\epsilon}$

$\begin{pmatrix} -\epsilon & -\epsilon \\ -\epsilon & -\epsilon \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow b = -a$

$|\alpha_{2\epsilon}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \underline{\frac{1}{\sqrt{2}}\left(|1\rangle - |2\rangle\right)}$

(c) $|\alpha(0)\rangle = |1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= c_1 |\alpha_0\rangle + c_2 |\alpha_{2\epsilon}\rangle$

$c_1 = \langle \alpha_0 | 1 \rangle = \frac{1}{\sqrt{2}} (1 \quad 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}$

$c_2 = \langle \alpha_{2\epsilon} | 1 \rangle = \frac{1}{\sqrt{2}} (1 \quad -1) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \frac{1}{\sqrt{2}}$

$\therefore \; |\alpha(0)\rangle = \frac{1}{\sqrt{2}} \left(|\alpha_0\rangle + |\alpha_{2\epsilon}\rangle \right)$

$\Rightarrow |\alpha(t)\rangle = \frac{1}{\sqrt{2}} \left(|\alpha_0\rangle e^{-i0t} + |\alpha_{2\epsilon}\rangle e^{-i\frac{2\epsilon}{\hbar}t} \right)$

$= \frac{1}{2} \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i\frac{2\epsilon}{\hbar}t} \right)$

$= \frac{1}{2} e^{-i\frac{\epsilon}{\hbar}t} \begin{pmatrix} e^{i\frac{\epsilon}{\hbar}t} + e^{-i\frac{\epsilon}{\hbar}t} \\ e^{i\frac{\epsilon}{\hbar}t} - e^{-i\frac{\epsilon}{\hbar}t} \end{pmatrix}$

$= e^{-i\frac{\epsilon}{\hbar}t} \begin{pmatrix} \cos(\frac{\epsilon}{\hbar}t) \\ i\sin(\frac{\epsilon}{\hbar}t) \end{pmatrix}$