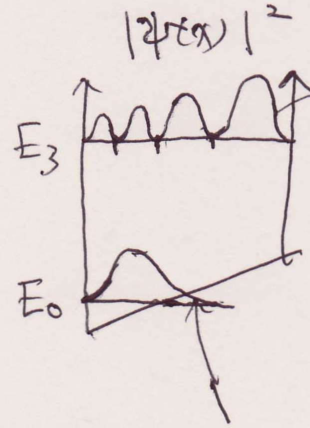
1.



* three nodes for E_3

* Both the spacing between nodes and the peak height increase as x approaches the right wall because the kinetic energy decreases

Penetration into the classically forbidden region

2.

(a) $k_1 = \frac{\sqrt{2mE}}{\hbar}$, $k_2 = \frac{\sqrt{2m(E+V_0)}}{\hbar}$

(b) $\psi_1(0) = \psi_2(0) \Rightarrow 1 + B = F$

$$\frac{d\psi_1(0)}{dx} = \frac{d\psi_2(0)}{dx} \Rightarrow 1 - B = \frac{k_2}{k_1} F$$

$$\Rightarrow 2 = \left(1 + \frac{k_2}{k_1}\right) F \Rightarrow F = \frac{2}{1+\frac{k_2}{k_1}} = \underline{\frac{2k_1}{k_1+k_2}}$$

$$B = F - 1 = \underline{\frac{k_1-k_2}{k_1+k_2}}$$

B is real

(c) $|\psi_1(x)|^2 = (1 + Be^{ik_1x})(1 + Be^{-ik_1x})$

$$= 1 + B^2 + B(e^{ik_1x} + e^{-ik_1x})$$

$$= 1 + B^2 + 2B\cos(k_1x)$$

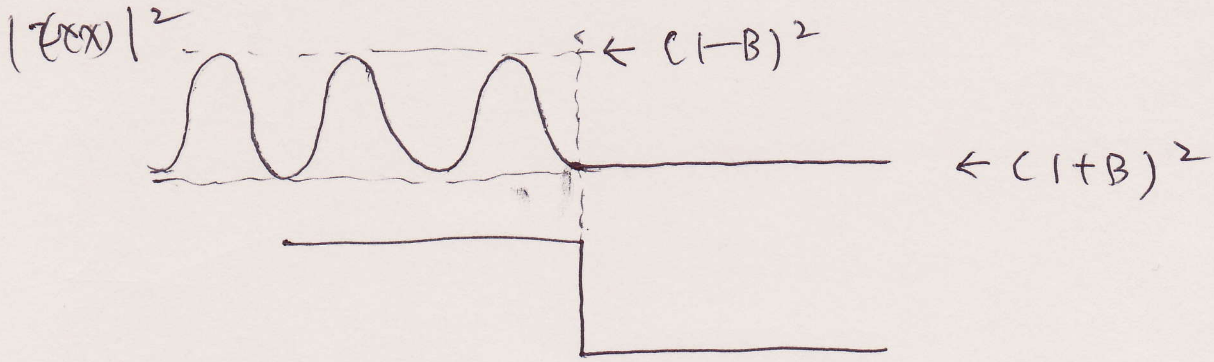
Because $k_2 > k_1$, $B < 0$.

Max of $|\psi_1(x)|^2 = 1 + B^2 - 2B = (1-B)^2$

Min of $|\psi_1(x)|^2 = 1 + B^2 + 2B = (1+B)^2$

$|\psi_2(x)|^2 = F^2 = (1+B)^2 =$ Min of $|\psi_1(x)|^2$

②



Standing wave is formed on $x<0$ because of the interference between the incoming wave (e^{ik_1x}) and the reflected wave (Be^{-ik_1x}).

3. (a) $\Psi(x,0) = \frac{1}{\sqrt{2}}(\psi_0(x) + \psi_1(x))$.

For this problem (harmonic oscillator), it is easier to use the operator method and the bra-ket Dirac notation.

$\Psi(x,0) \Rightarrow |\Psi(t=0)\rangle$

$\psi_0(x) \Rightarrow |0\rangle$, $\psi_1(x) \Rightarrow |1\rangle$.

Then the given state can be written as

$$|\Psi(t=0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$\rightarrow \underline{|\Psi(t)\rangle} = \frac{1}{\sqrt{2}}\left(|0\rangle e^{-i\frac{E_0}{\hbar}t} + |1\rangle e^{-i\frac{E_1}{\hbar}t}\right)$$

$$= \underline{\frac{1}{\sqrt{2}}\left[|0\rangle e^{-i\frac{\omega}{2}t} + |1\rangle e^{-i\frac{3\omega}{2}t}\right]}$$

(b) $\langle x \rangle \equiv \langle \Psi(t)|x|\Psi(t)\rangle$

Use $x \equiv \sqrt{\frac{\hbar}{2m\omega}}(a^{+}+a)$

$\cancel{(a^\dagger + a)\,(|0\rangle e^{-}}$

$$(a^\dagger + a)\,|\Psi(t)\rangle = \frac{1}{\sqrt{2}}(a^\dagger + a)\left[\,|0\rangle e^{-i\frac{\omega}{2}t} + |1\rangle e^{-i\frac{3\omega}{2}t}\right]$$

$$\left[\; a^\dagger|0\rangle = |1\rangle, \quad a|0\rangle = 0, \quad a^\dagger|1\rangle = \sqrt{2}\,|2\rangle, \quad a|1\rangle = |0\rangle \;\right]$$

$$= \frac{1}{\sqrt{2}}\left(|1\rangle e^{-i\frac{\omega t}{2}} + |0\rangle e^{-i\frac{3\omega t}{2}} + \sqrt{2}\,|2\rangle e^{-i\frac{3\omega t}{2}}\right).$$

Since $\langle\Psi(t)| = \frac{1}{\sqrt{2}}\left(\langle 0|e^{i\frac{\omega}{2}t} + \langle 1|e^{i\frac{3\omega}{2}t}\right)$

$$\langle\Psi(t)|\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)|\Psi(t)\rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \times \frac{1}{2} \times \left[\left(\langle 0|e^{i\frac{\omega t}{2}} + \langle 1|e^{i\frac{3\omega}{2}t}\right)\right.$$

$$\left.\times\left(|1\rangle e^{-i\frac{\omega t}{2}} + |0\rangle e^{-i\frac{3\omega t}{2}} + \sqrt{2}\,|2\rangle e^{-i\frac{3\omega}{2}t}\right)\right].$$

$\langle 0|1\rangle = 0$

$\langle 0|2\rangle = 0$

$\langle 1|2\rangle = 0$

$\langle 1|1\rangle = 1$

$\langle 0|0\rangle = 1$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t} + e^{i\omega t}\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\cos\omega t$$

4. $Y(\theta,\phi) = A\sin\theta\cos\phi$

(a)

$$= \frac{A}{2}\sin\theta\left(e^{i\phi} + e^{-i\phi}\right)$$

$$= \frac{A}{2}\left(-\left(\frac{8\pi}{3}\right)^{\frac{1}{2}} Y_1^{1} + \left(\frac{8\pi}{3}\right)^{\frac{1}{2}} Y_1^{-1}\right)$$

We use the Dirac notation,

$$Y(\theta,\varphi) \Rightarrow |Y\rangle, \quad Y_1^{\pm 1} \Rightarrow |Y_1^{\pm 1}\rangle$$

~~Since $\langle Y|Y\rangle = 1$~~

$$|Y\rangle = -A\left(\frac{2\pi}{3}\right)^{\frac{1}{2}}\left(|Y_1^1\rangle - |Y_1^{-1}\rangle\right)$$

Since $\langle Y|Y\rangle = 1$, $\langle Y_1^{-1}|Y_1^1\rangle = 0$, $\langle Y_1^1|Y_1^1\rangle = 1$, $\langle Y_1^{-1}|Y_1^{-1}\rangle = 1$

$$\Rightarrow \underset{1}{\langle Y|Y\rangle} = A^2 \cdot \frac{2\pi}{3} \cdot \left((\langle Y_1^1| - \langle Y_1^{-1}|)(|Y_1^1\rangle - |Y_1^{-1}\rangle)\right)$$

$$= A^2 \frac{2\pi}{3} \cdot \left(\langle Y_1^1|Y_1^1\rangle + \langle Y_1^{-1}|Y_1^{-1}\rangle \right)$$

$$= A^2 \frac{4\pi}{3}$$

$$\Rightarrow A = \sqrt{\frac{3}{4\pi}}$$

$$|Y\rangle = -\left(\frac{3}{4\pi}\right)^{\frac{1}{2}}\left(\frac{2\pi}{3}\right)^{\frac{1}{2}}\left(|Y_1^1\rangle - |Y_1^{-1}\rangle\right)$$

$$= -\frac{1}{\sqrt{2}}\left(|Y_1^1\rangle - |Y_1^{-1}\rangle\right)$$

(b) $$L_z|Y\rangle = -\frac{1}{\sqrt{2}}\left(\hbar|Y_1^1\rangle + \hbar|Y_1^{-1}\rangle\right) = -\frac{\hbar}{\sqrt{2}}\left(|Y_1^1\rangle + |Y_1^{-1}\rangle\right)$$

「 $L_z|Y_1^1\rangle = \hbar m|Y_1^1\rangle = \hbar|Y_1^1\rangle$

$L_z|Y_1^{-1}\rangle = -\hbar|Y_1^{-1}\rangle$ 」

$$\Rightarrow \langle L_z\rangle = \langle Y|L_z|Y\rangle = \frac{\hbar}{2}\left((\langle Y_1^1| - \langle Y_1^{-1}|)(|Y_1^1\rangle + |Y_1^{-1}\rangle)\right)$$

$$= \frac{\hbar}{2}\left(\langle Y_1^1|Y_1^1\rangle - \langle Y_1^{-1}|Y_1^{-1}\rangle\right)$$

$$= 0$$

$$\left\lceil L^2 |Y_l^m\rangle = \hbar^2 l(l+1) |Y_l^m\rangle \right\rfloor$$

So $L^2|Y\rangle = -\frac{\hbar^2}{\sqrt{2}}\left(1\times(1+1)|Y_1^1\rangle - 1\times(1+1)|Y_1^{-1}\rangle\right)$

$$= -\sqrt{2}\hbar^2\left(|Y_1^1\rangle - |Y_1^{-1}\rangle\right)$$

$$= -2\hbar^2 \frac{|Y_1^1\rangle - |Y_1^{-1}\rangle}{\sqrt{2}} = 2\hbar^2|Y\rangle$$

$\Rightarrow \langle Y|L^2|Y\rangle = \langle Y|2\hbar^2|Y\rangle = \underline{2\hbar^2}$

5. (a) $H = -\gamma B_0 S_z = -\frac{\hbar}{2}\gamma B_0 \sigma_z = \underline{-\frac{\hbar}{2}\gamma B_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$

(b) $S_x \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = b \Rightarrow \underline{\chi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

(c) Since $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the energy eigenspinor for $E_1 = -\frac{\hbar}{2}\gamma B_0$

and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is that for $E_2 = \frac{\hbar}{2}\gamma B_0$

$$\chi(t) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\frac{E_1}{\hbar}t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\frac{E_2}{\hbar}t}\right]$$

$$\underline{= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{+i\frac{\gamma B_0}{2}t} \\ e^{-i\frac{\gamma B_0}{2}t} \end{pmatrix}}$$

(d)

$$\langle S_x \rangle = \langle \chi(t) | S_x | \chi(t) \rangle$$

$$= \frac{1}{\sqrt{2}} \left(e^{-i\frac{\gamma B_0 t}{2}},\ e^{i\frac{\gamma B_0}{2}t} \right) \cdot \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\gamma B_0}{2}t} \\ e^{-i\frac{\gamma B_0}{2}t} \end{pmatrix}$$

$$= \frac{\hbar}{4} \left(e^{-i\frac{\gamma B_0 t}{2}},\ e^{i\frac{\gamma B_0 t}{2}} \right) \begin{pmatrix} e^{-i\frac{\gamma B_0}{2}t} \\ e^{i\frac{\gamma B_0}{2}t} \end{pmatrix}$$

$$= \frac{\hbar}{4} \left(e^{-i\gamma B_0 t} + e^{i\gamma B_0 t} \right)$$

$$= \underline{\frac{\hbar}{2} \cos(\gamma B_0 t)}$$