+++ Formula Sheet +++

$x = \sqrt{\dfrac{\hbar}{2\, m\omega}}\,(a^\dagger + a)$, $p = i\sqrt{\dfrac{\hbar m\omega}{2}}\,(a^\dagger - a)$, $a^\dagger|n\rangle = \sqrt{n+1}\;|n+1\rangle$ and $a|n\rangle = \sqrt{n}\;|n-1\rangle$

$H = \hbar\omega\left(a^\dagger a + \dfrac{1}{2}\right)$, $[a, a^\dagger] = 1$

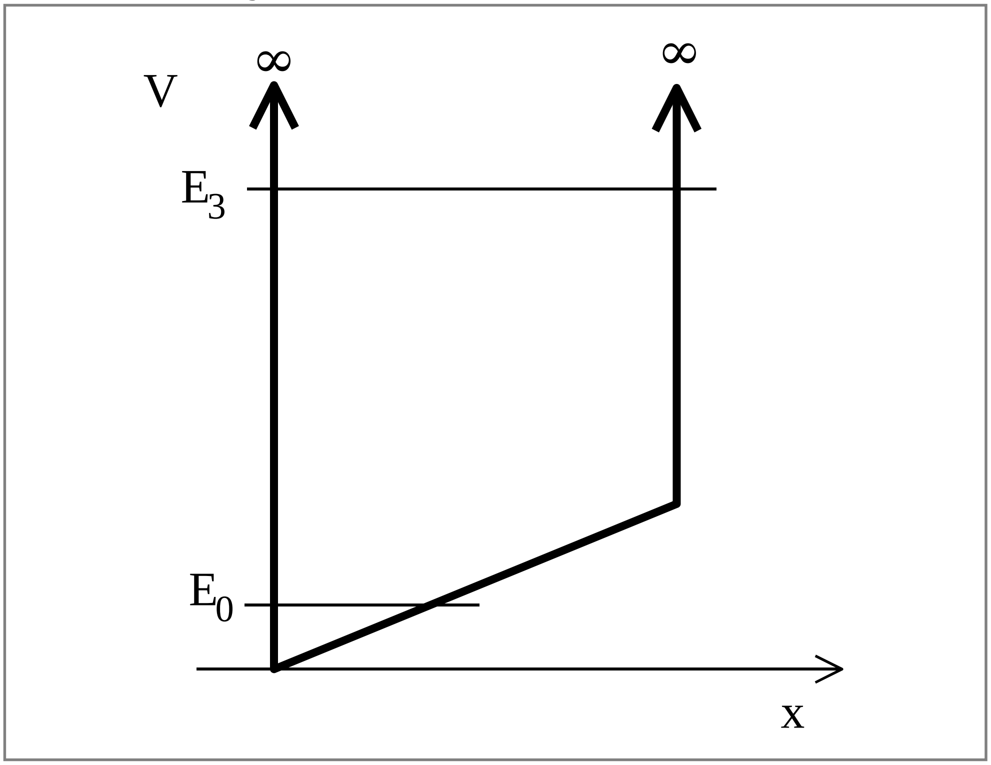
$L_\pm = L_x \pm iL_y$, and $L_\pm|l, m\rangle = \hbar\sqrt{l(l+1) - m(m\pm 1)}\;|l, m\pm 1\rangle$

$J(x,t) = \dfrac{i\hbar}{2\,m}\left(\Psi\dfrac{\partial\Psi^*}{\partial x} - \Psi^*\dfrac{\partial\Psi}{\partial x}\right)$

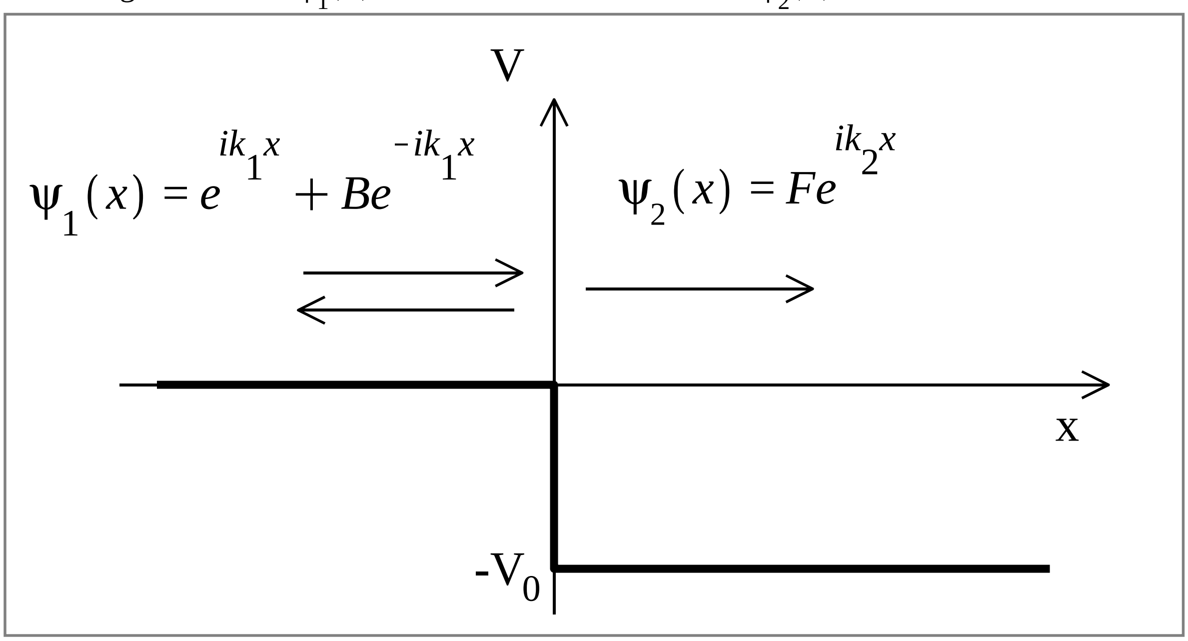
**Table 4.2:** The first few spherical harmonics, $Y_l^m(\theta, \phi)$.

| | |
|---|---|
| $Y_0^0 = \left(\frac{1}{4\pi}\right)^{1/2}$ | $Y_2^{\pm 2} = \left(\frac{15}{32\pi}\right)^{1/2} \sin^2\theta e^{\pm 2i\phi}$ |
| $Y_1^0 = \left(\frac{3}{4\pi}\right)^{1/2} \cos\theta$ | $Y_3^0 = \left(\frac{7}{16\pi}\right)^{1/2} (5\cos^3\theta - 3\cos\theta)$ |
| $Y_1^{\pm 1} = \mp\left(\frac{3}{8\pi}\right)^{1/2} \sin\theta e^{\pm i\phi}$ | $Y_3^{\pm 1} = \mp\left(\frac{21}{64\pi}\right)^{1/2} \sin\theta(5\cos^2\theta - 1)e^{\pm i\phi}$ |
| $Y_2^0 = \left(\frac{5}{16\pi}\right)^{1/2} (3\cos^2\theta - 1)$ | $Y_3^{\pm 2} = \left(\frac{105}{32\pi}\right)^{1/2} \sin^2\theta\cos\theta e^{\pm 2i\phi}$ |
| $Y_2^{\pm 1} = \mp\left(\frac{15}{8\pi}\right)^{1/2} \sin\theta\cos\theta e^{\pm i\phi}$ | $Y_3^{\pm 3} = \mp\left(\frac{35}{64\pi}\right)^{1/2} \sin^3\theta e^{\pm 3i\phi}$ |

1. (15 pts) Consider a particle in an infinite well shown below. Without solving the Schroedinger equation, sketch the probability density $|\psi(x)|^2$ of the eigenfunctions corresponding to the ground state ($E_0$) and the third excited state ($E_3$), and explain all the key qualitative features.




2. A particle of mass $m$ and kinetic energy $E > 0$ approaches an abrupt potential drop $V_0$ as shown below. As you know, in this problem, the wave function cannot be normalized, but the the relative amplitudes are still meaningful. Assume $\psi_1(x) = e^{ik_1x} + Be^{-ik_1x}$ for $x < 0$ and $\psi_2(x) = Fe^{ik_2x}$ for $x > 0$.




(a) (2 pts) Express $k_1$ and $k_2$ in terms of $E$, $V_0$ and $m$.

(b) (8 pts) By applying appropriate boundary conditions at $x = 0$, obtain $B$ and $F$ in terms of $k_1$ and $k_2$ .

(c) (10 pts) Sketch the probability density $|\psi(x)|^2$ ( $|\psi_1(x)|^2$ for $x < 0$ and $|\psi_2(x)|^2$ for $x > 0$) over the entire region.

What are the maximum and minimum values of $|\psi_1(x)|^2$ and how do they compare with $|\psi_2(x)|^2$?

(d) (5 pts) Obtain the transmission coefficient $T$ in terms of $k_1$ and $k_2$.

3. A particle in the harmonic oscillator potential $\left(V = \frac{1}{2}m\omega^2x^2\right)$ starts out in the normalized wavefunction

$$\Psi(x, 0) = \frac{1}{\sqrt{2}}\left[\psi_0(x) + \psi_1(x)\right],$$

where $\psi_0(x)$ and $\psi_1(x)$ are the ground state and the first excited state eigenfunction, repectively.

(a) (5 points) Construct $\Psi(x, \mathrm{t})$.

(b) (10 points) Find the expectation value $\langle x\rangle$ at time t of this state.

4. Angular wavefunction of a particle is given by $Y(\theta, \phi) = A\sin(\theta)\cos(\phi)$ (Hint: Express it by a sum of spherical harmonics)

(a) (8 points) Find the normalization constant $A$.

(b) (12 points) Evaluate the expectation values $\langle L_z\rangle$ and $\langle L^2\rangle$ of this state.

5. A spin-1/2 particle at rest in a uniform magnetic field pointing in the z-direction is described by the Hamiltonian:

$$H = -\gamma B_0 S_z.$$

(a) (5 points) Write down the matrix describing this Hamiltonian: our basis is the standard $\left|m = \frac{1}{2}\right\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $\left|m = -\frac{1}{2}\right\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.

(b) (5 points) If, at t = 0, measurement of $S_x$ resulted in $\frac{\hbar}{2}$, what is the spinor $\chi$(t=0) of the state right after the measurement?

(Do not forget to normalize.)

(c) (5 points) At a later time t (>0), what is the corresponding spinor $\chi$(t)?

(d) (10 points) Evaluate $\langle S_x\rangle$ at time t.