HW 9 Solution

Note Title

4.32 (a)

From $\chi(t) = \begin{pmatrix} \cos(\frac{\alpha}{2}) e^{i\gamma B_0 t/2} \\ \sin(\frac{\alpha}{2}) e^{-i\gamma B_0 t/2} \end{pmatrix}$

and $\chi_+^{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\Rightarrow P(S_x \Rightarrow \frac{\hbar}{2}) = |\langle \chi_+^{(x)} | \chi \rangle|^2$$

$$= \frac{1}{2} \left| (1, 1) \begin{pmatrix} \cos(\frac{\alpha}{2}) e^{i\gamma B_0 t/2} \\ \sin(\frac{\alpha}{2}) e^{-i\gamma B_0 t/2} \end{pmatrix} \right|^2$$

$$= \frac{1}{2} \left| \cos(\frac{\alpha}{2}) e^{i\gamma B_0 t/2} + \sin(\frac{\alpha}{2}) e^{-i\gamma B_0 t/2} \right|^2$$

$e^{i\gamma B_0 t}$

$$= \frac{1}{2} \Big[\cos^2(\frac{\alpha}{2}) + \sin^2(\frac{\alpha}{2}) + \cos(\frac{\alpha}{2}) \sin(\frac{\alpha}{2}) + \cos(\frac{\alpha}{2}) \sin(\frac{\alpha}{2}) e^{-i\gamma B_0 t} \Big]$$

$$= \frac{1}{2} \Big[1 + 2\cos(\frac{\alpha}{2}) \sin(\frac{\alpha}{2}) \cos(\gamma B_0 t) \Big]$$

$$= \underline{\frac{1}{2} \Big[1 + \sin(\alpha) \cos(\gamma B_0 t) \Big]}$$

(b) $P(S_y \Rightarrow \frac{\hbar}{2}) = |\langle \chi_+^{(y)} | \chi \rangle|^2$

$$= \frac{1}{2} \left| (1, -i) \begin{pmatrix} \cos(\frac{\alpha}{2}) e^{i\gamma B_0 t/2} \\ \sin(\frac{\alpha}{2}) e^{-i\gamma B_0 t/2} \end{pmatrix} \right|^2$$

$$= \frac{1}{2}\left| \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2} - i \sin\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t/2} \right|^2$$

$$= \frac{1}{2}\Big[\cos^2\left(\frac{\alpha}{2}\right) + \sin^2\left(\frac{\alpha}{2}\right) + i \cos\left(\frac{\alpha}{2}\right) \sin\left(\frac{\alpha}{2}\right) \cdot e^{i\gamma B_0 t}$$
$$- i \cos\left(\frac{\alpha}{2}\right) \sin\left(\frac{\alpha}{2}\right) e^{-i\gamma B_0 t} \Big]$$

$$= \frac{1}{2}\left[1 - 2\cos\left(\frac{\alpha}{2}\right) \sin\left(\frac{\alpha}{2}\right) \sin(\gamma B_0 t) \right]$$

$$= \underline{\frac{1}{2}\left[1 - \sin(\alpha) \sin(\gamma B_0 t) \right]}$$

(c) $P\left(S_z \Rightarrow \frac{\hbar}{2}\right) = \left| \langle \chi_+^{(z)} | \chi \rangle \right|^2$

$$= \left| \cos\left(\frac{\alpha}{2}\right) e^{i\gamma B_0 t/2} \right|^2$$
$$= \underline{\cos^2\left(\frac{\alpha}{2}\right)}$$

4.34 (a) $S_- |1\,0\rangle = \left(S_-^{(1)} + S_-^{(2)}\right) \frac{\uparrow\downarrow + \downarrow\uparrow}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2}}\Big[(S_-^{(1)} \uparrow)\downarrow + \downarrow(S_-^{(2)}\uparrow)$$
$$+ \cancel{(S_-^{(1)} \downarrow)}\uparrow + \uparrow \cancel{(S_-^{(2)} \downarrow)} \Big]$$

$$= \frac{\hbar}{\sqrt{2}}\left[\downarrow\downarrow + \downarrow\downarrow \right]$$

$$= \sqrt{2}\hbar \downarrow\downarrow = \underline{\sqrt{2}\hbar\, |1\,-1\rangle}$$

(b) $S_-|0\,0\rangle = (S_-^{(1)} + S_-^{(2)}) \dfrac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}$

$\underset{\uparrow}{=} \dfrac{\hbar}{\sqrt{2}} (\downarrow\downarrow - \downarrow\downarrow) = 0$

Just as above,

$S_+|0\,0\rangle = (S_+^{(1)} + S_+^{(2)}) \dfrac{\uparrow\downarrow - \downarrow\uparrow}{\sqrt{2}}$

$\underset{\uparrow}{=} \dfrac{\hbar}{\sqrt{2}} (\uparrow\uparrow - \uparrow\uparrow) = 0$

Similarly as above

(c) $S^2 = S^{(1)^2} + S^{(2)^2} + 2S^{(1)} \cdot S^{(2)}$ ← Eq 4.179

← Eq 4.142

$S^{(1)^2} \uparrow\uparrow = \frac{3}{4}\hbar^2 \uparrow\uparrow$

$S^{(2)^2} \uparrow\uparrow = \frac{3\hbar^2}{4} \uparrow\uparrow$

$S^{(1)} \cdot S^{(2)} \uparrow\uparrow = (S_x^{(1)}\uparrow)(S_x^{(2)}\uparrow) + (S_y^{(1)}\uparrow)(S_y^{(2)}\uparrow)$

$+ (S_z^{(1)}\uparrow)(S_z^{(2)}\uparrow)$ — $(*)$

$S_x^{(1)}\uparrow = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}\downarrow$

$S_y^{(1)}\uparrow = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}0\\i\end{pmatrix} = i\frac{\hbar}{2}\downarrow$

$S_z^{(1)}\uparrow = \frac{\hbar}{2}\uparrow$

Thus, $\circledast = \frac{\hbar^2}{4}(\downarrow\downarrow + i^2\downarrow\downarrow + \uparrow\uparrow)$

$= \frac{\hbar^2}{4}\uparrow\uparrow$

$\therefore\ S^2|1\,1\rangle = S^2\uparrow\uparrow = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + \frac{2\hbar^2}{4}\right)\uparrow\uparrow$

$= 2\hbar^2\uparrow\uparrow$

$= 1(1+1)\hbar^2|1\,1\rangle$

$|1\,-1\rangle = \downarrow\downarrow$ and if we replace $\uparrow\uparrow$ by $\downarrow\downarrow$ above,

$S_x^{(1)}\downarrow = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\hbar}{2}\uparrow$

$S_y^{(1)}\downarrow = \frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = -i\frac{\hbar}{2}\uparrow$

$S_z^{(1)}\downarrow = \frac{\hbar}{2}\downarrow$

$S^{(1)}\cdot S^{(2)}\downarrow\downarrow = \frac{\hbar^2}{4}(\uparrow\uparrow + i^2\uparrow\uparrow + \downarrow\downarrow)$

$= \frac{\hbar^2}{4}\downarrow\downarrow$

Then

$S^2|1\,-1\rangle = S^2\downarrow\downarrow = \left(\frac{3\hbar^2}{4} + \frac{3\hbar^2}{4} + \frac{2\hbar^2}{4}\right)\downarrow\downarrow$

$= 1(1+1)\hbar^2|1\,-1\rangle$

Therefore both $\uparrow\uparrow$ and $\downarrow\downarrow$ are the eigenstates of S^2 with the correct eigenvalue of $2\hbar^2$.

Prob. 4.35

(a) Addition of two spin $\frac{1}{2}$ particles can give either 1 or 0. Addition of another spin $\frac{1}{2}$ can yield $1+\frac{1}{2}$ or $1-\frac{1}{2}$ or $0+\frac{1}{2}$. Thus baryons can have spins of $\underline{\frac{3}{2}}$ or $\underline{\frac{1}{2}}$

(b) Addition of two spin $\frac{1}{2}$ particles can yield spin of $\underline{1}$ or $\underline{0}$

4.49 $\quad \chi = A \begin{pmatrix} 1-2i \\ 2 \end{pmatrix}$

(a) $\langle \chi | \chi \rangle = 1 \Rightarrow |A|^2 (1+2i, 2) \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} = 1$

$\Rightarrow |A|^2 (1+4+4) = 1$

$\Rightarrow A = \frac{1}{3}$

(b) $P(S_z \Rightarrow \frac{\hbar}{2}) = |\langle \chi_+^{(z)} | \chi \rangle|^2$

$= \frac{1}{9} |1-2i|^2 = \frac{5}{9}$

$P(S_z \Rightarrow -\frac{\hbar}{2}) = |\langle \chi_-^{(z)} | \chi \rangle|^2$

$= \frac{1}{9} (2)^2 = \frac{4}{9}$

$\langle S_z \rangle = \frac{5}{9} \cdot \frac{\hbar}{2} + \frac{4}{9} \left(-\frac{\hbar}{2}\right) = \frac{\hbar}{18}$

(c) $P(S_x \Rightarrow \frac{\hbar}{2}) = |\langle \chi_+^{(x)} | \chi \rangle|^2$

$= \frac{1}{9} \left| \frac{1}{\sqrt{2}} (1, 1) \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2$

$= \frac{1}{18} |1-2i+2|^2 = \frac{9+4}{18} = \frac{13}{18}$

$$P(S_x \Rightarrow -\frac{\hbar}{2}) = \frac{1}{18}\left| (1, -1) \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2$$

$$= \frac{1}{18} \left| 1-2i-2 \right|^2 = \frac{4+1}{18} = \underline{\frac{5}{18}}$$

$$\langle S_x \rangle = \frac{13}{18}\left(\frac{\hbar}{2}\right) + \frac{5}{18}\left(-\frac{\hbar}{2}\right)$$

$$= \frac{8}{18} \cdot \frac{\hbar}{2} = \underline{\frac{2}{9}\hbar}$$

(d) $$P(S_y \Rightarrow \frac{\hbar}{2}) = \left| \langle S_+^{(y)} | \chi \rangle \right|^2$$

$$= \frac{1}{9} \left| \frac{1}{\sqrt{2}} (1, -i) \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2$$

From Prob. 4.29 above

$$= \frac{1}{18} \left| 1-2i-2i \right|^2 = \frac{16+1}{18}$$

$$= \underline{\frac{17}{18}}$$

$$P(S_y \Rightarrow -\frac{\hbar}{2}) = \left| \langle S_-^{(y)} | \chi \rangle \right|^2$$

$$= \frac{1}{9} \left| \frac{1}{\sqrt{2}} (1, i) \begin{pmatrix} 1-2i \\ 2 \end{pmatrix} \right|^2$$

$$= \frac{1}{18} \left| 1-2i+2i \right|^2 = \underline{\frac{1}{18}}$$

$$\langle S_y \rangle = \frac{17}{18} \frac{\hbar}{2} + \frac{1}{18}\left(-\frac{\hbar}{2}\right) = \underline{\frac{4}{9}\hbar}$$

Prob. 5.4

(a) $\psi_\pm(r_1, r_2) = A\left[\psi_a(r_1)\psi_b(r_2) \pm \psi_b(r_1)\psi_a(r_2)\right]$

$\Rightarrow \langle\psi_\pm|\psi_\pm\rangle = 1$ gives

$$1 = A^2\left[\langle\psi_a(r_1)|\langle\psi_b(r_2)| \pm \langle\psi_b(r_1)|\langle\psi_a(r_2)|\right]$$
$$\times\left[|\psi_a(r_1)\rangle|\psi_b(r_2)\rangle \pm |\psi_b(r_1)\rangle|\psi_a(r_2)\rangle\right]$$
$$= A^2\left[\langle\psi_a(r_1)|\psi_a(r_1)\rangle\langle\psi_b(r_2)|\psi_b(r_2)\rangle\right.$$
$$\pm\langle\psi_a(r_1)|\psi_b(r_1)\rangle\langle\psi_b(r_2)|\psi_a(r_2)\rangle$$
$$\pm\langle\psi_b(r_1)|\psi_a(r_1)\rangle\langle\psi_a(r_2)|\psi_b(r_2)\rangle$$
$$\left.+\langle\psi_b(r_1)|\psi_b(r_1)\rangle\langle\psi_a(r_2)|\psi_a(r_2)\rangle\right]$$
$$= A^2\left[1 \pm 0 \pm 0 + 1\right] = 2A^2 \Rightarrow \underline{A = \frac{1}{\sqrt{2}}}$$

(b) $\psi_+(r_1, r_2) = 2A\left(\psi_a(r_1)\psi_a(r_2)\right)$

$$\Rightarrow \langle\psi_+|\psi_+\rangle = 1 = 4A^2\langle\psi_a(r_1)|\psi_a(r_1)\rangle$$
$$\cdot\langle\psi_a(r_2)|\psi_a(r_2)\rangle$$
$$= 4A^2$$
$$\Rightarrow \underline{A = \frac{1}{2}}$$