# HW 8 solution

Note Title 11/12/2010

4.23 $Y_2^1(\theta,\varphi) = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\varphi}$

$L_+ Y_2^1 = \hbar\sqrt{2\cdot 3 - 1\cdot 2}\; Y_2^2 = 2\hbar\, Y_2^2$

$\Rightarrow Y_2^2 = \frac{1}{2\hbar} L_+ Y_2^1 = \frac{1}{2} e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right)$

$\times \left(-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\varphi}\right)$

$= -\frac{1}{2}\sqrt{\frac{15}{8\pi}}\, e^{i\varphi}\Big[\cos^2\theta\, e^{i\varphi} - \sin^2\theta\, e^{i\varphi}$

$+ i\cot\theta\sin\theta\cos\theta \cdot i\; e^{i\varphi}\Big]$

$= -\frac{1}{2}\sqrt{\frac{15}{8\pi}}\, e^{2i\varphi}\left[\cos^2\theta - \sin^2\theta - \cos^2\theta\right]$

$= \underline{\sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{2i\varphi}}$

4.24 (a) $H = \frac{1}{2}I\omega^2 = \frac{1}{2}I\left(\frac{L}{I}\right)^2 = \frac{L^2}{2I}$

$\left[L = I\omega \Rightarrow \omega = \frac{L}{I}\right]$

$= \frac{L^2}{ma^2}$

$\uparrow$

$I = 2m\left(\frac{a}{2}\right)^2 = \frac{ma^2}{2}$

$\therefore\; E_n = \frac{\hbar^2 n(n+1)}{ma^2}, \quad n = 0, 1, 2, \ldots$

(b) $\underline{Y_n^m(\theta,\varphi)}$

There are $2n+1$ possible values for $m$. So the degeneracy is $\boxed{2n+1}$

4.29 (a) $S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$$S_y \chi = \frac{\hbar}{2}\lambda\chi \quad , \quad \chi = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} -\lambda & -i \\ i & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + i^2 = 0 \Rightarrow (\lambda-1)(\lambda+1) = 0$$

$$\Rightarrow \lambda = \pm 1$$

$\therefore$ eigenvalues ; $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

For $\frac{\hbar}{2}$

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow ai - b = 0 \Rightarrow b = ai$$

$$\Rightarrow \chi_+^{(y)} = \begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ i \end{pmatrix} = \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}$$

For $-\frac{\hbar}{2}$

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow b = -ai$$

$$\Rightarrow \chi_-^{(y)} = a\begin{pmatrix} 1 \\ -i \end{pmatrix} = \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

(b) Measurement of $S_y$ yields one of its eigenvalues, $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$

$$P(S_y \Rightarrow \tfrac{\hbar}{2}) = |\langle \chi_+^{(y)} | \chi \rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1, -i) \begin{pmatrix} a \\ b \end{pmatrix} \right|^2$$

$$= \underline{\frac{1}{2} | a - bi |^2}$$

$$P(S_y \Rightarrow -\tfrac{\hbar}{2}) = |\langle \chi_-^{(y)} | \chi \rangle|^2$$

$$= \left| \frac{1}{\sqrt{2}} (1, i) \begin{pmatrix} a \\ b \end{pmatrix} \right|^2 = \underline{\frac{1}{2} | a + bi |^2}$$

$$P(S_y \Rightarrow \tfrac{\hbar}{2}) + P(S_y = -\tfrac{\hbar}{2}) = \frac{1}{2}|a - bi|^2 + \frac{1}{2}|a + bi|^2$$

$$= \frac{1}{2}\left[ (a - bi)(a^* + b^* i) + (a + bi)(a^* - b^* i) \right]$$

$$= \frac{1}{2}\Big[ |a|^2 + |b|^2 + ab^* i - a^* b i$$
$$+ |a|^2 + |b|^2 - ab^* i + a^* b i \Big]$$

$$= |a|^2 + |b|^2 = 1$$ due to normalization

(c) $$S_y^2 = \frac{\hbar^2}{4} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$= \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_y^2 \chi = \lambda \chi$$

$$\Rightarrow \begin{vmatrix} \frac{\hbar^2}{4} - \lambda & 0 \\ 0 & \frac{\hbar^2}{4} - \lambda \end{vmatrix} = 0$$

$$\Rightarrow \left(\lambda - \frac{\hbar^2}{4}\right)^2 = 0 \Rightarrow \lambda = \frac{\hbar^2}{4}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$\Rightarrow$ any arbitrary $\begin{pmatrix} a \\ b \end{pmatrix}$ is an eigenstate of $S_y^2$.

Measurement of $S_y^2$ always yields $\frac{\hbar^2}{4}$ with 100% probability

4.31 $S_z$ for spin 1 particle should have three eigenstates: In $|s\ m\rangle$ notation, they should be $|1\ 1\rangle$, $|1\ 0\rangle$, $|1\ -1\rangle$.

If we define $|1\rangle \equiv |1\ 1\rangle$, $|2\rangle \equiv |1\ 0\rangle$ and $|3\rangle \equiv |1\ -1\rangle$, we should have

$$S_z|1\rangle = \hbar\,|1\rangle$$
$$S_z|2\rangle = 0\,|2\rangle$$
$$S_z|3\rangle = -\hbar\,|3\rangle$$

Because $(S_z)_{ij} = \langle i | S_z | j \rangle$,

$\langle 1 | S_z | 1 \rangle = \hbar$ , $\langle 2 | S_z | 2 \rangle = 0$
$\langle 3 | S_z | 3 \rangle = -\hbar$ and all other $(S_z)_{ij}$
are zero because $\langle i | j \rangle = \delta_{ij}$

$$\Rightarrow S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Now from $S_\pm | 1\ m \rangle = \hbar \sqrt{1 \cdot 2 - m(m\pm 1)}\, | 1\ m\pm 1 \rangle$

, $S_+ | 1 \rangle \equiv S_+ | 1, 1 \rangle = 0$
$S_+ | 2 \rangle \equiv S_+ | 1, 0 \rangle = \hbar \sqrt{2} | 1, 1 \rangle = \sqrt{2}\hbar | 1 \rangle$
$S_+ | 3 \rangle \equiv S_+ | 1, -1 \rangle = \hbar \sqrt{2} | 1, 0 \rangle = \sqrt{2}\hbar | 2 \rangle$

$S_- | 1 \rangle \equiv S_- | 1, 1 \rangle = \sqrt{2}\hbar | 1, 0 \rangle = \sqrt{2}\hbar$
$S_- | 2 \rangle \equiv S_- | 1, 0 \rangle = \sqrt{2}\hbar | 1, -1 \rangle = \sqrt{2}\hbar | 3 \rangle$
$S_- | 3 \rangle \equiv S_- | 1, -1 \rangle = 0$

$\Rightarrow \langle 1 | S_+ | 1 \rangle = 0$, $\langle 1 | S_+ | 2 \rangle = \sqrt{2}\hbar$, $\langle 1 | S_+ | 3 \rangle = 0$
$\langle 2 | S_+ | 1 \rangle = 0$, $\langle 2 | S_+ | 2 \rangle = 0$ , $\langle 2 | S_+ | 3 \rangle = \sqrt{2}\hbar$
$\langle 3 | S_+ | 1 \rangle = 0$, $\langle 3 | S_+ | 2 \rangle = 0$ , $\langle 3 | S_+ | 2 \rangle = 0$

$$\Rightarrow S_+ = \sqrt{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\langle 1|S_-|1\rangle = 0, \langle 1|S_-|2\rangle = 0, \langle 1|S_-|3\rangle = 0$

$\langle 2|S_-|1\rangle = \sqrt{2}\hbar, \langle 2|S_-|2\rangle = 0, \langle 2|S_-|3\rangle = 0$

$\langle 3|S_-|1\rangle = 0, \langle 3|S_-|2\rangle = \sqrt{2}\hbar, \langle 3|S_-|3\rangle = 0$

$$\Rightarrow S_- = \sqrt{2}\hbar \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{\sqrt{2}i} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

$$= \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$