HW 6 solution

Note Title

G3.12

$$\langle x\rangle = \int_x \Psi^*(x,t)\, x\, \Psi(x,t)\, dx$$

$$= \frac{1}{2\pi\hbar}\int_x \Big[\Big[\int_{p'} e^{ip'x/\hbar}\,\Phi(p',t)\,dp'\Big]^*$$

$$\cdot\, x \int_p e^{ipx/\hbar}\,\Phi(p,t)\,dp\Big]\, dx$$

Now our objetive is to get rid $\underbrace{\text{of the}}_{dp}$ "x" integral.

$$x\int_p e^{ipx/\hbar}\,\Phi(p,t)\,dp = \int_p -i\hbar\frac{d}{dp}\left(e^{ipx/\hbar}\right)\Phi(p,t)$$

$$= -i\hbar\Big[\cancelto{0}{e^{ipx/\hbar}\,\Phi(p,t)\Big|_{p=-\infty}^{p=\infty}} - \int_p e^{ipx/\hbar}\frac{\partial\Phi(p,t)}{\partial p}\,dp$$

$$= i\hbar\int_p e^{ipx/\hbar}\frac{\partial}{\partial p}\Phi(p,t)\,dp$$

Now

$$\langle x\rangle = \frac{i\hbar}{2\pi\hbar}\int_{p'}\int_p \Phi^*(p',t)\frac{\partial}{\partial p}\Phi(p,t)$$

$$\times \int_x e^{-ip'x/\hbar}\cdot e^{ipx/\hbar}\,dx\,dp\,dp'$$

Since $\delta\left(\frac{p-p'}{\hbar}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\frac{(p-p')}{\hbar}x}\,dx$

$$\langle x\rangle = i\int_{p'}\int_p \Phi^*(p',t)\frac{\partial}{\partial p}\Phi(p,t)\,\delta\left(\frac{p-p'}{\hbar}\right)dp\,dp'.$$

since $\delta\left(\frac{p-p'}{\hbar}\right) = \hbar\,\delta(p-p')$,

$$\langle x\rangle = i\hbar \int_p \Phi^*(p,t)\frac{\partial}{\partial p}\Phi(p,t)\,dp$$

$$= \int_p \Phi^*(p,t)\left(-\frac{\hbar}{i}\frac{\partial}{\partial p}\right)\Phi(p,t)\,dp \, .$$

G3.13

(a) $[AB, C] = ABC - CAB$

$$= ABC - ACB + ACB - CAB$$

$$= A(BC - CB) + (AC - CA)B$$

$$= A[B, C] + [A, C]B$$

(b) $[x^n, p]f(x) = \left[x^n, \frac{\hbar}{i}\frac{d}{dx}\right]f(x)$

$$= x^n \frac{\hbar}{i}\frac{d}{dx}f(x) - \frac{\hbar}{i}\frac{d}{dx}\left(x^n f(x)\right)$$

$$= \frac{\hbar}{i}\left[\cancel{x^n\frac{d}{dx}f(x)} - n x^{n-1} f(x) - \cancel{x^n \frac{d}{dx}f(x)}\right]$$

$$= -\frac{\hbar}{i} n x^{n-1} f(x)$$

$$= i\hbar n x^{n-1} f(x)$$

$$\therefore [x^n, p] = i\hbar n x^{n-1}$$

(C) $[f(x), p]\, g(x) = [f(x), \frac{\hbar}{i}\frac{d}{dx}]\, g(x)$

$$= \frac{\hbar}{i}\left\{ f(x)\frac{d}{dx}g(x) - \frac{d}{dx}(f(x)g(x)) \right\}$$

$$= \frac{\hbar}{i}\left\{ \cancel{f(x)\frac{d}{dx}g(x)} - \frac{d}{dx}f(x)\cdot g(x) - \cancel{f(x)\frac{d}{dx}g(x)} \right\}$$

$$= i\hbar\frac{d}{dx}f(x)\cdot g(x)$$

$$\therefore \quad [f(x), p] = i\hbar\frac{d}{dx}f(x)$$

G3.17 (a) $Q = 1$

$$\frac{d}{dt}\langle 1\rangle = \frac{i}{\hbar}\langle[\hat{H}, 1]\rangle = 0$$

$\langle 1\rangle = \langle\Psi|\Psi\rangle$. Thus this is equivalent to Eq. (1.27).

(b) $Q = H$, $\frac{d\langle H\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H}, \hat{H}]\rangle = 0$

This is just the energ conservation as we have seen in Eq. 2.39.

(c) $Q = x$, $\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H}, x]\rangle$

$$[\hat{H}, x] = [\frac{p^2}{2m} + V(x), x]$$

$$= \frac{1}{2m}[p^2, x]$$

$$= \frac{1}{2m}\left\{ \underbrace{p[p,x]}_{-i\hbar p} + \underbrace{[p,x]p}_{-i\hbar p} \right\}$$

$\uparrow$
using Prob. 3.13

$$= \frac{\hbar}{mi}\, p$$

Thus $$\frac{d\langle x\rangle}{dt} = \frac{i}{\hbar}\,\frac{\hbar}{mi}\,\langle p\rangle$$

$$= \frac{\langle p\rangle}{m}$$

This is equivalent to Eq. 1.33

(d) $Q = p$

$$\frac{d\langle p\rangle}{dt} = \frac{i}{\hbar}\langle [H, p]\rangle$$

$$[H, p] = \left[\frac{p^2}{2m} + V(x), p\right] = [V(x), p]$$

$$= i\hbar \frac{dV(x)}{dx}$$ using Eq 3.65.

Thus $$\frac{d\langle p\rangle}{dt} = -\left\langle \frac{dV}{dx}\right\rangle$$

This is the Ehrenfest's theorem in Eq. 1.38

G3.23

$$\hat{H} = \epsilon \left(|1\rangle\langle 1| - |2\rangle\langle 2| + |1\rangle\langle 2| + |2\rangle\langle 1| \right)$$

with $|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $|2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|1\rangle\langle 1| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1 \; 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$|2\rangle\langle 2| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (0 \; 1) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|1\rangle\langle 2| = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (0 \; 1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$|2\rangle\langle 1| = \begin{pmatrix} 0 \\ 1 \end{pmatrix} (1 \; 0) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Thus, $\hat{H} = \underline{\begin{pmatrix} \epsilon & \epsilon \\ \epsilon & -\epsilon \end{pmatrix}}$

$\hat{H}|\alpha\rangle = \lambda|\alpha\rangle$, $|\alpha\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$

$$\Rightarrow \begin{vmatrix} \epsilon - \lambda & \epsilon \\ \epsilon & -\epsilon - \lambda \end{vmatrix} = 0$$

$$\Rightarrow (\epsilon - \lambda)(-\epsilon - \lambda) - \epsilon^2 = 0$$

$$\Rightarrow \lambda^2 - \epsilon^2 - \epsilon^2 = 0 \Rightarrow \lambda = \pm\sqrt{2}\,\epsilon$$

For $\lambda = \underline{\sqrt{2}\,\epsilon}$

$$\begin{pmatrix} \epsilon - \sqrt{2}\epsilon & \epsilon \\ \epsilon & -\epsilon - \sqrt{2}\epsilon \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$\Rightarrow (1-\sqrt{2})\,a_1 + a_2 = 0$

$\Rightarrow a_2 = (\sqrt{2}-1)\,a_1$

$\Rightarrow |\alpha\rangle = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} a_1 = \underline{a_1\left(|1\rangle + (\sqrt{2}-1)\,|2\rangle\right)}$

For $\lambda = \underline{-\sqrt{2}\,\epsilon}$

$(1+\sqrt{2})\,a_1 + a_2 = 0 \Rightarrow a_2 = -(\sqrt{2}+1)\,a_1$

$\Rightarrow |\alpha\rangle = \begin{pmatrix} 1 \\ -(\sqrt{2}+1) \end{pmatrix} a_1 = \underline{a_1\left(|1\rangle - (\sqrt{2}+1)\,|2\rangle\right)}$

3.27 (a) The state collapses to the eigenstate corresponding to a_1, which is $\underline{\psi_1}$

(b) $P_{b_1} = |\langle \phi_1 | \psi_1 \rangle|^2 = \frac{9}{25}$

$P_{b_2} = |\langle \phi_2 | \psi_1 \rangle|^2 = \frac{16}{25}$

(c) $P_{a_1} = P_{b_1} \to P_{a_1} + P_{b_2} \to P_{a_1}$

$P_{b_1} \to P_{a_1} = |\langle \phi_1 | \psi_1 \rangle|^2 \cdot |\langle \psi_1 | \phi_1 \rangle|^2$

$P_{b_2} \to P_{a_1} = |\langle \phi_2 | \psi_1 \rangle|^2 \cdot |\langle \psi_1 | \phi_2 \rangle|^2$

Considering that $|\langle \alpha | \beta \rangle|^2 = |\langle \beta | \alpha \rangle|^2$,

$P_{b_1} \to P_{a_1} = |\langle \phi_1 | \psi_1 \rangle|^4$

$P_{b_2} \to P_{a_1} = |\langle \phi_2 | \psi_1 \rangle|^4$

$\therefore P_{a_1} = \frac{9}{25} \cdot \frac{9}{25} + \frac{16}{25} \cdot \frac{16}{25} = \frac{337}{625}$

$= 0.5392$

$\boxed{3.37}$ $H = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ b & 0 & a \end{pmatrix}$

Let's solve for eigenenergies and eigenstates of this Hamiltonian.

$H|\alpha\rangle = \lambda |\alpha\rangle, \quad |\alpha\rangle = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$0 = \begin{vmatrix} a-\lambda & 0 & b \\ 0 & c-\lambda & 0 \\ b & 0 & a-\lambda \end{vmatrix}$$

$$= (a-\lambda)(c-\lambda)(a-\lambda) - b \cdot b(c-\lambda)$$

$$= (c-\lambda)\left((a-\lambda)^2 - b^2\right)$$

$$= (c-\lambda)(\lambda-a-b)(\lambda-a+b)$$

$$\Rightarrow \lambda = c, \; a+b, \; a-b$$

For $\lambda = a \pm b$,

$$\begin{pmatrix} \mp b & 0 & b \\ 0 & c-a\mp b & 0 \\ b & 0 & \mp b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Rightarrow \quad \mp b x + b z = 0 \quad \Rightarrow \quad z = \pm x$$

$$(c-a\mp b)\, y = 0 \quad \Rightarrow \quad y = 0$$

$$\Rightarrow |\alpha\rangle_{a\pm b} = \begin{pmatrix} x \\ 0 \\ \pm x \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \pm 1 \end{pmatrix}$$

$$|\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = |\alpha_c\rangle$$

Thus $|\mathcal{S}(t)\rangle = e^{-i\frac{c}{\hbar}t}\, |\alpha_c\rangle$

$$= e^{-i\frac{c}{\hbar}t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

(b) $$|\mathcal{S}(0)\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = c_c\,|\alpha_c\rangle + c_{a+b}\,|\alpha_{a+b}\rangle + c_{a-b}\,|\alpha_{a-b}\rangle$$

$$\Rightarrow c_c = \langle \alpha_c | \mathcal{S}(0)\rangle = (0\ 1\ 0)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = 0$$

$$c_{a\pm b} = \langle \alpha_{a\pm b} | \mathcal{S}(0)\rangle = \frac{1}{\sqrt{2}}(1\ 0\ \pm 1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \pm\frac{1}{\sqrt{2}}$$

Thus

$$|\mathcal{S}(t)\rangle = \frac{1}{\sqrt{2}}|\alpha_{a+b}\rangle e^{-i\frac{a+b}{\hbar}t} - \frac{1}{\sqrt{2}}|\alpha_{a-b}\rangle e^{-i\frac{a-b}{\hbar}t}$$

$$= \frac{1}{2}e^{-i\frac{a}{\hbar}t}\left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^{-i\frac{b}{\hbar}t} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{+i\frac{b}{\hbar}t}\right]$$

$$= \frac{1}{2}e^{-i\frac{a}{\hbar}t}\begin{pmatrix} e^{-i\frac{b}{\hbar}t} - e^{i\frac{b}{\hbar}t} \\ 0 \\ e^{-i\frac{b}{\hbar}t} + e^{i\frac{b}{\hbar}t} \end{pmatrix}$$

$$= e^{-i\frac{a}{\hbar}t}\begin{pmatrix} -i\sin\left(\frac{b}{\hbar}t\right) \\ 0 \\ \cos\left(\frac{b}{\hbar}t\right) \end{pmatrix}$$