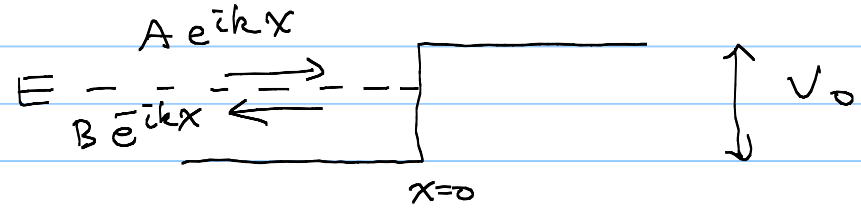
HW 4 solution

Note Title

1.



$E < V_0$

(a)

$x<0$

$$-\frac{\hbar^2}{2m}\psi''(x) = E\psi(x)$$

$$\Rightarrow \quad \psi''(x) = -\frac{2mE}{\hbar^2}\psi(x) = -k^2\psi(x)$$

$$\psi(x) = Ae^{ikx} + Be^{-ikx}, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$x>0$

$$-\frac{\hbar^2}{2m}\psi''(x) + V_0\psi(x) = E\psi(x)$$

$$\Rightarrow \quad \psi''(x) = \underbrace{\frac{2m}{\hbar^2}(V_0-E)}_{\kappa^2}\psi(x) \quad , \kappa = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$$\Rightarrow \psi(x) = Fe^{-\kappa x} + Ge^{\kappa x}$$

$$\psi(x\to\infty) = 0 \Rightarrow G = 0$$

Now boundary conditions at $x=0$

① $\psi(x=0-) = \psi(x=0+)$

$$\Rightarrow \quad A + B = F$$

② $\psi'(x=0-) = \psi'(x=0+)$

$$\Rightarrow \quad ik(A-B) = -\kappa F \Rightarrow A - B = \frac{i\kappa}{k}F$$

$$\Rightarrow \quad 2A = \left(1+\frac{i\kappa}{k}\right)F$$

$$\Rightarrow F = \frac{2}{1+\frac{i\kappa}{k}}A$$

$$B = F - A = \frac{1-\frac{i\kappa}{k}}{1+\frac{i\kappa}{k}}A$$

Thus

$$|\Psi(x,t)|^2 = |\psi(x)e^{-i\frac{E}{\hbar}t}|^2 = |\psi(x)|^2$$

$\boxed{\text{For } x<0}$

$$|\psi(x)|^2 = |A|^2 \left| e^{ikx} + \frac{B}{A} e^{-ikx} \right|^2$$
$$= |A|^2 \left(1 + \left|\frac{B}{A}\right|^2 + \left(\frac{B}{A}\right)^* e^{2ikx} + \left(\frac{B}{A}\right) e^{-2ikx} \right)$$

$$\frac{B}{A} = \frac{1-\frac{i\kappa}{k}}{1+\frac{i\kappa}{k}} = \frac{\left(1-\frac{i\kappa}{k}\right)^2}{1+\left(\frac{\kappa}{k}\right)^2} = \frac{1-\left(\frac{\kappa}{k}\right)^2 - 2i\frac{\kappa}{k}}{1+\left(\frac{\kappa}{k}\right)^2}$$

$$\left[\left|\frac{B}{A}\right| = \frac{1+\left(\frac{\kappa}{k}\right)^2}{1+\left(\frac{\kappa}{k}\right)^2} = 1\right]$$

Thus $\frac{B}{A} = e^{i\theta}$

with $\cos\theta = \frac{1-\left(\frac{\kappa}{k}\right)^2}{1+\left(\frac{\kappa}{k}\right)^2}$

$$|\psi(x)|^2 = |A|^2 \left(1 + 1 + e^{-i\theta} e^{2ikx} + e^{i\theta} e^{-2ikx} \right)$$
$$= |A|^2 \left(2 + 2\cos(2kx-\theta) \right)$$

$$= \begin{cases} \text{max} = 4|A|^2 \\ \text{min} = 0 \\ \text{at } x=0, \quad = 2|A|^2 (1+\cos\theta) \end{cases}$$

$$= 2|A|^2 \left(1 + \frac{1-\left(\frac{\kappa}{k}\right)^2}{1+\left(\frac{\kappa}{k}\right)^2} \right)$$

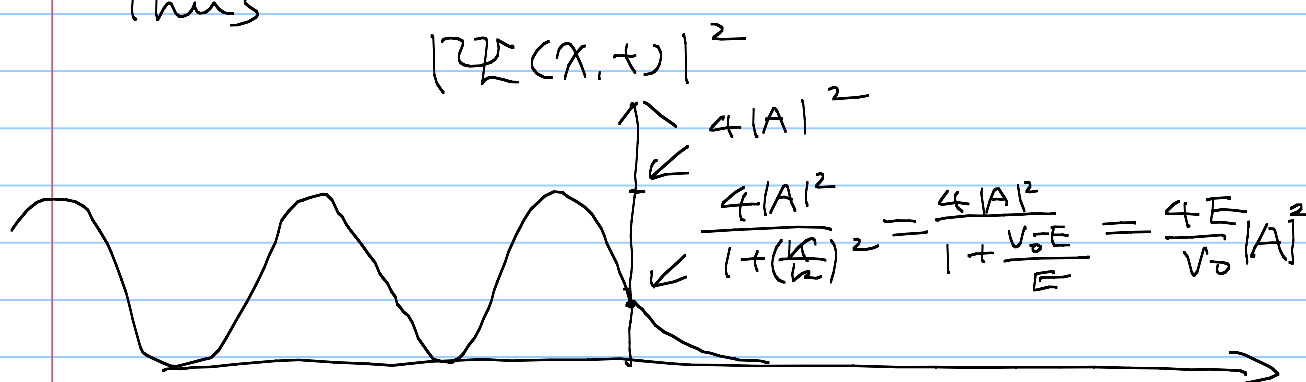
$$= \frac{4|A|^2}{1+\left(\frac{\kappa}{k}\right)^2}$$

For $x>0$

$$|\psi(x)|^2 = |F|^2 e^{-2\kappa x} = \frac{4}{1+\frac{\kappa^2}{k^2}} e^{-2\kappa x} \cdot |A|^2$$

$$= \begin{cases} \text{at } x=0, \quad = \frac{4|A|^2}{1+\left(\frac{\kappa}{k}\right)^2} \\ \text{at } x=\infty, \quad = 0 \end{cases}$$

Thus



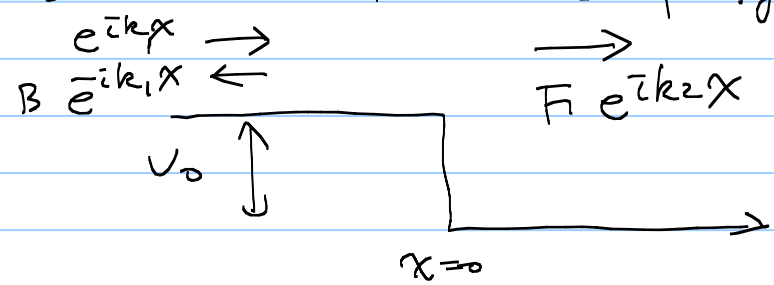
(b) $R = \left|\frac{B}{A}\right|^2 = 1$

$T = 1 - R = 0$, for any $E < V_0$

(c) $\dfrac{|\Psi(x=0,t)|^2}{|A|^2} \approx \begin{cases} 0 & \text{for } \frac{E}{V_0} \ll 1 \\ 4 & \text{for } \frac{E}{V_0} \approx 1 \end{cases}$

2. Let's assume $A = 1$ to simplify the algebra

(a)



As we did in class, $k_1 = \dfrac{\sqrt{2m(E-V_0)}}{\hbar}$

$k_2 = \dfrac{\sqrt{2mE}}{\hbar}$

and $k_1 < k_2$.

Now, boundary conditions require

① $\psi(0-) = \psi(0+)$

$1 + B = F$

② $\psi'(0-) = \psi'(0+)$

$ik_1(1-B) = ik_2 F$

$\Rightarrow 1 - B = \dfrac{k_2}{k_1} F$

$\Rightarrow 2 = \left(1 + \dfrac{k_2}{k_1}\right) F \Rightarrow F = \dfrac{2k_1}{k_1 + k_2}$

$B = F - 1 = \dfrac{2}{1 + \frac{k_2}{k_1}} - 1 = \dfrac{k_1 - k_2}{k_1 + k_2}$

For $x < 0$

$|\psi(x)|^2 = |1 + Be^{-ik_1 x}|^2 = 1 + B^2 + B(e^{ik_1 x} + e^{-ik_1 x})$

$= 1 + B^2 + 2B\cos k_1 x$

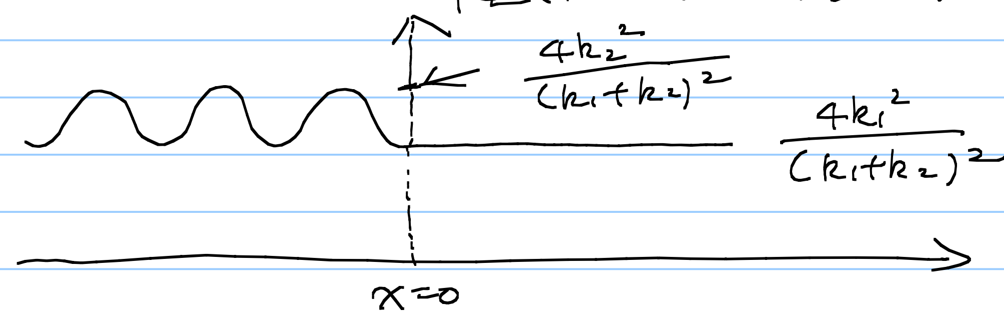
$$= \begin{cases} \max, & (1-B)^2 = \left(\frac{k_2}{k_1}\right)^2 F^2 = \frac{4k_2^2}{(k_1+k_2)^2} \\ \min, & (1+B)^2 = F^2 = \frac{4k_1^2}{(k_1+k_2)^2} \end{cases}$$

For $x > 0$

$$|\psi(x)|^2 = |F|^2 = \frac{4k_1^2}{(k_1+k_2)^2}.$$

Because $k_1 < k_2$,

$$|\Psi(x,t)|^2 = |\psi(x)|^2$$



(b)

$$R = \left|\frac{B}{A}\right|^2 = |B|^2 = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2$$

$$= \left(\frac{\sqrt{E-V_0} - \sqrt{E}}{\sqrt{E-V_0} + \sqrt{E}}\right)^2$$

$$T = \frac{k_2|F|^2}{k_1|A|^2} = \frac{k_2}{k_1}|F|^2 = \frac{k_2}{k_1}\cdot\frac{4k_1^2}{(k_1+k_2)^2}$$

$$= \frac{4k_1k_2}{(k_1+k_2)^2} = \frac{4\sqrt{(E-V_0)E}}{(\sqrt{E-V_0}+\sqrt{E})^2}$$

3. Prob 2.26

Fourier transform of $\delta(x)$ is

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ik(0)} = \frac{1}{\sqrt{2\pi}}$$

So $\delta(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) e^{ikx} dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} dx$

4. Prob. 2.37

$$\Psi(x,0) = A\sin^3(\pi x/a) \qquad 0 \le x \le a$$

$$\sin^3 x = \left(\frac{e^{ix} - e^{-ix}}{2i}\right)^3 = \frac{e^{3ix} - 3e^{ix} + 3e^{-ix} - e^{-3ix}}{-8i}$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$= -\frac{1}{4}\left(\frac{e^{3ix} - e^{-3ix}}{2i}\right) + \frac{3}{4}\left(\frac{e^{ix} - e^{-ix}}{2i}\right)$$

$$= -\frac{1}{4}\sin 3x + \frac{3}{4}\sin x$$

Thus $\Psi(x,0) = \frac{A}{4}\left(3\sin\left(\frac{\pi x}{a}\right) - \sin\left(\frac{3\pi x}{a}\right)\right)$

$$= \frac{A}{4}\left(3\sqrt{\frac{a}{2}}\,\psi_1(x) - \sqrt{\frac{2}{a}}\,\psi_3(x)\right)$$

$$= \frac{A}{4}\sqrt{\frac{a}{2}}\left(3\psi_1(x) - \psi_3(x)\right)$$

Normalization requires

$$1 = \left(3\frac{A}{4}\sqrt{\frac{a}{2}}\right)^2 + \left(\frac{A}{4}\sqrt{\frac{a}{2}}\right)^2$$

$$= A^2\cdot\left(\frac{9}{16}\cdot\frac{a}{2} + \frac{1}{16}\frac{a}{2}\right) = A^2\frac{\overset{5}{\cancel{10}}\,a}{\cancel{32}\,16}$$

$$\Rightarrow \underline{A = \sqrt{\frac{16}{5a}} = \frac{4}{\sqrt{5a}}}$$

Now $\Psi(x,0) = \frac{1}{\sqrt{5a}}\cdot\sqrt{\frac{a}{2}}\left(3\psi_1(x) - \psi_3(x)\right)$

$$= \frac{1}{\sqrt{10}}\left(3\psi_1(x) - \psi_3(x)\right)$$

Thus

$$\Psi(x,t) = \frac{1}{\sqrt{10}}\left(3\psi_1(x)e^{-i\frac{E_1}{\hbar}t} - \psi_3(x)e^{-i\frac{E_3 t}{\hbar}}\right)$$

$$|\Psi(x,t)|^2 = \frac{1}{10}\Big(9\psi_1^2 + \psi_3^2 - 3\psi_1\psi_3 e^{-i\frac{E_3-E_1}{\hbar}t}$$

$$-3\psi_3\psi_1 e^{i\frac{E_3-E_1}{\hbar}t}\Big)$$

$$= \frac{1}{10}\left(9\psi_1^2 + \psi_3^2 - 6\psi_1\psi_3\cos\left(\frac{E_3 - E_1}{\hbar}t\right)\right)$$

Thus

$$\langle x\rangle = \int_0^a x\,|\Psi(x,t)|^2 dx = \frac{9}{10}\int_0^a x\psi_1^2 dx$$

$$+ \frac{1}{10}\int_0^a x\psi_3^2 dx - \frac{3}{5}\left[\int_0^a x\psi_1\psi_3 dx\right]\cos(\quad)$$

Due to symmetry, $\int_0^a x\,\psi_n^2(x)\,dx = \frac{a}{2}$

for any "n".

$$\int_0^a x\psi_1\psi_3 dx = \frac{2}{a}\int_0^a x\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{3\pi}{a}x\right)dx$$

$$= \frac{2}{a}\cdot\frac{a}{\pi}\int_0^\pi \frac{a}{\pi}y\sin(y)\sin(3y)\,dy$$

$\frac{\pi}{a}x = y$

$dx = \frac{a}{\pi}dy$

$$= \frac{2a}{\pi^2}\int_0^\pi \frac{1}{2}y\left[\cos(2y) - \cos(4y)\right]dy$$

$$= \frac{a}{\pi^2}\left[y\left[\frac{\sin(2y)}{2} - \frac{\sin(4y)}{4}\right] + \left[\frac{\cos(2y)}{4} - \frac{\cos 4y}{16}\right]\right]_0^\pi$$

$$= 0$$

$$\therefore \langle x\rangle = \frac{9}{10}\left(\frac{a}{2}\right) + \frac{1}{10}\left(\frac{a}{2}\right) - 0$$

$$= \frac{a}{2} \quad \text{independent of time}$$

$$\langle E\rangle = P_{E_1}E_1 + P_{E_3}E_3$$

$$= \frac{9}{10}\frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 + \frac{1}{10}\frac{\hbar^2}{2m}\left(\frac{3\pi}{a}\right)^2$$

$$= \frac{\hbar^2}{20m}\left(\frac{\pi}{a}\right)^2(9+9) = \frac{9\hbar^2}{10m}\left(\frac{\pi}{a}\right)^2$$

5.

(a) $\Psi(x,0) = \frac{1}{\sqrt{2a}}\left(\theta(x+a) - \theta(x-a)\right)$

(b) $\langle p \rangle = \frac{\hbar}{i} \int_{-\infty}^{\infty} \Psi^*(x,0) \frac{d}{dx} \Psi(x,0)\, dx$

$$= \frac{\hbar}{i} \cdot \frac{1}{2a} \int_{-\infty}^{\infty} \left[\theta(x+a) - \theta(x-a)\right] \times \left[\delta(x+a) - \delta(x-a)\right] dx$$

$$= \frac{\hbar}{i}\frac{1}{2a}\left[\theta(0) - \theta(-2a) - \theta(2a) + \theta(0)\right]$$

$$= \frac{\hbar}{i}\frac{1}{2a}\left[\frac{1}{2} - 0 - 1 + \frac{1}{2}\right] = \underline{0}$$

$$\langle p^2 \rangle = -\hbar^2 \frac{1}{2a} \int_{-\infty}^{\infty} \left[\theta(x+a) - \theta(x-a)\right] \cdot \left[\delta'(x+a) - \delta'(x-a)\right] dx$$

$$= -\frac{\hbar^2}{2a}\left\{-\left[\theta'(0) - \theta'(-2a)\right] + \left[\theta'(2a) - \theta'(0)\right]\right\}$$

$$= -\frac{\hbar^2}{2a}\left\{-2\delta(0) + \cancel{\delta(-2a)}^{\,0} + \cancel{\delta(2a)}^{\,0}\right\}$$

$$= \frac{\hbar^2}{a}\delta(0) = \underline{\infty}$$

(c) $\langle x \rangle = \frac{1}{2a}\int_{-a}^{a} x\, dx = \underline{0}$

$\because$ odd integrand

$$\langle x^2 \rangle = \frac{1}{2a}\int_{-a}^{a} x^2\, dx = \frac{1}{2a}\cdot\frac{2a^3}{3} = \underline{\frac{a^2}{3}}$$

(d) $\sigma_x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \frac{a}{\sqrt{3}}$

$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \infty$

$\therefore \sigma_p \cdot \sigma_x = \infty > \frac{\hbar}{2}$

uncertainty principle holds.