Prob 2.12

$$\langle x\rangle_n = \sqrt{\frac{\hbar^2}{2m\omega}}\ \langle a_+ + a_-\rangle_n = 0$$

$\langle p\rangle_n = 0$ for any stationary states

$$\langle x^2\rangle_n = \frac{\hbar}{2m\omega}\ \langle a_+^2 + a_+a_- + a_-a_+ + a_-^2\rangle_n$$

$$= \frac{\hbar}{2m\omega}\ \langle a_+a_- + a_-a_+\rangle_n$$

$$= \frac{\hbar}{2m\omega}\ \langle 2a_+a_- + 1\rangle_n$$

$$= \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right)$$

$$\langle p^2\rangle_n = -\frac{\hbar m\omega}{2}\ \langle a_+^2 - a_+a_- - a_-a_+ + a_-^2\rangle_n$$

$$= \frac{\hbar m\omega}{2}\ \langle a_+a_- + a_-a_+\rangle_n$$

$$= \frac{\hbar m\omega}{2}\cdot(2n+1)$$

$$= \hbar m\omega\left(n+\frac{1}{2}\right)$$

$$\sigma_x \cdot \sigma_p = \sqrt{\langle p^2\rangle_n}\sqrt{\langle x^2\rangle_n} = \hbar\left(n+\frac{1}{2}\right) \geq \frac{\hbar}{2}$$

$$\langle T\rangle = \left\langle \frac{p^2}{2m}\right\rangle_n = \frac{1}{2m}\hbar m\omega\left(n+\frac{1}{2}\right)$$

$$= \frac{1}{2}\hbar\omega\left(n+\frac{1}{2}\right) = \frac{E_n}{2}$$

Prob 2.13

(a) $1 = \int |\Psi(x,0)|^2 dx = A^2(9+16)$

$$\Rightarrow A = \frac{1}{5}$$

(b) $\Psi(x,t) = \frac{1}{5}\left[3\psi_0(x)e^{-i\frac{\omega}{2}t} + 4\psi_1(x)e^{-i\frac{3\omega}{2}t}\right]$

$$|\Psi(x,t)|^2 = \frac{1}{25}\left(9|\psi_0|^2 + 16|\psi_1|^2 + 12\psi_0^*\psi_1 e^{-i\omega t} + 12\psi_0\psi_1^* e^{i\omega t}\right)$$

$$= \frac{1}{25}\left(9\psi_0^2 + 16\psi_1^2 + 24\psi_0\psi_1 \cos(\omega t)\right)$$

(c) $\langle x\rangle = \int_{-\infty}^{\infty} x|\Psi(x,t)|^2 dx$ odd integrand

$$= \frac{1}{25}\left[\cancel{\int_{-\infty}^{\infty} 9x\psi_0^2 dx}^{\,0} + \cancel{\int_{-\infty}^{\infty} 16x\psi_1^2 dx}^{\,0} + 24\int_{-\infty}^{\infty} x\psi_0\psi_1 \cos(\omega t)\, dx\right]$$

$$\int \psi_0 x \psi_1\, dx = \sqrt{\frac{\hbar}{2m\omega}}\int \psi_0 (a_+ + a_-)\psi_1\, dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\int \psi_0 (a_-\psi_1)\, dx = \sqrt{\frac{\hbar}{2m\omega}}\int \psi_0^2 dx$$

$$= \sqrt{\frac{\hbar}{2m\omega}}$$

Thus $\underbrace{\langle x\rangle = \frac{24}{25}\cdot\sqrt{\frac{\hbar}{2m\omega}}\cos(\omega t)}$

$\langle p\rangle = m\frac{d\langle x\rangle}{dt} = \underbrace{-\frac{24}{25}\sqrt{\frac{m\omega\hbar}{2}}\sin(\omega t)}$

If ψ_1 is replaced by ψ_2, $\int_{-\infty}^{\infty} x\psi_0\psi_2\, dx$ will be zero because the integrand is odd. Therefore both $\langle x\rangle$ and $\langle p\rangle$ will remain zero, although $|\Psi(x,t)|^2$ do vibrate symmetrically around $x=0$ because ψ_0 and ψ_2 are both even ftns.

Now let's check if the Ehrenfest's theorem

$$\frac{d\langle p\rangle}{dt} = \left\langle -\frac{\partial V}{\partial x}\right\rangle \quad \text{holds.}$$

$$\frac{d\langle p\rangle}{dt} = -\frac{24}{25}\cdot\omega\sqrt{\frac{m\omega\hbar}{2}}\cos(\omega t) \qquad \cos(\omega t)$$

$$\left\langle -\frac{\partial V}{\partial x}\right\rangle = \left\langle -\frac{d}{dx}\left(\frac{1}{2}m\omega^2x^2\right)\right\rangle = -m\omega^2\langle x\rangle = -\frac{24}{25}\omega\sqrt{\frac{m\omega\hbar}{2}}$$

$\therefore$ Ehrenfest's theorem holds.

(d) $\underset{\frac{1}{2}\hbar\omega}{P_{E_0}} = \frac{9}{25}$, $\underset{\frac{3}{2}\hbar\omega}{P_{E_1}} = \frac{16}{25}$

Prob. 2.14

The new ground state energy is now

$$\frac{\hbar\omega'}{2} = \hbar\omega.$$

So measuring an energy value of $\frac{\hbar\omega}{2}$, which is less than the ground state energy, is impossible. That is the probability is zero.

$$P_{E=\hbar\omega} = \left|\int \psi^*_{\substack{new\\n=0}}\cdot\Psi(x,0)\,dx\right|^2$$

$$= \left|\int \psi^*_{\substack{\omega'=2\omega\\n=0}}(x)\,\psi_{\substack{\omega\\n=0}}(x)\,dx\right|^2$$

$$= \left(\frac{2m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{2}}\left[\int e^{-\frac{m\omega}{\hbar}x^2}\cdot e^{-\frac{m\omega}{2\hbar}x^2}\,dx\right]^2$$

$$= \frac{\sqrt{2}\,m\omega}{\pi\hbar}\left[\int_{-\infty}^{\infty} e^{-\frac{3m\omega}{2\hbar}x^2}\,dx\right]^2$$

$$= \frac{\sqrt{2}\,m\omega}{\pi\hbar}\cdot\left(\sqrt{\frac{2\hbar}{3m\omega}}\cdot\sqrt{\pi}\right)^2 = \frac{\sqrt{2}\,\cancel{m\omega}}{\cancel{\pi}\cancel{\hbar}}\,\frac{2\cancel{\pi}\cancel{\hbar}}{3\cancel{m\omega}}$$

$$= \frac{2\sqrt{2}}{3} = 0.943$$

Prob. 2.21

$$\Psi(x,0) = A e^{-a|x|}$$

(a)

$$1 = \int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx = 2|A|^2 \int_0^{\infty} e^{-2ax} dx$$

$$= 2|A|^2 \frac{1}{2a} \Rightarrow A = \sqrt{a}$$

(b) $$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Psi(x,0) e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \sqrt{a} \int_{-\infty}^{\infty} e^{-a|x|} e^{-ikx} dx$$

$$= \sqrt{\frac{a}{2\pi}} \left[\int_0^{\infty} e^{-x(a+ik)} dx + \int_{-\infty}^{0} e^{x(a-ik)} dx \right]$$

$$= \sqrt{\frac{a}{2\pi}} \left[-\frac{e^{-x(a+ik)}}{a+ik} \Big|_0^{\infty} + \frac{e^{x(a-ik)}}{a-ik} \Big|_{-\infty}^{0} \right]$$

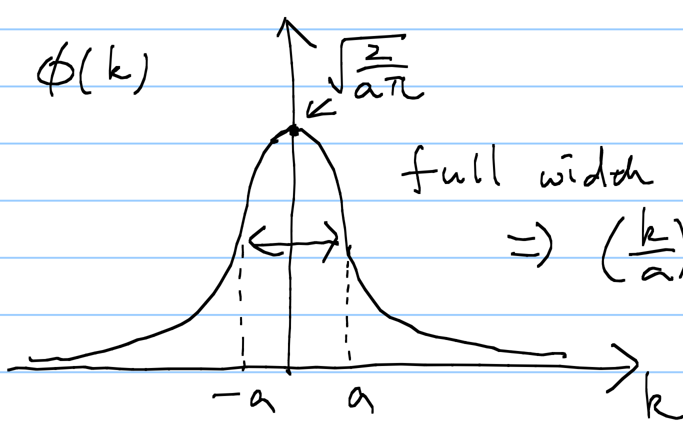
$$= \sqrt{\frac{a}{2\pi}} \left(\frac{1}{a+ik} + \frac{1}{a-ik} \right] = \sqrt{\frac{2a}{\pi}} \frac{a}{a^2+k^2}$$

$$= \underline{\sqrt{\frac{2}{a\pi}} \cdot \frac{1}{1+\left(\frac{k}{a}\right)^2}}$$

(c) $$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \phi(k) e^{i(kx - \frac{\hbar k^2}{2m}t)} dk$$

$$= \frac{1}{\pi\sqrt{a}} \int_{-\infty}^{\infty} \frac{1}{1+\left(\frac{k}{a}\right)^2} e^{i(kx - \frac{\hbar k^2}{2m}t)} dk$$

(d) $\phi(k)$



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$$\Rightarrow \left(\frac{k}{a}\right)^2 = 1 \Rightarrow k = a$$

$$\therefore FWHM = 2a$$

$\boxed{\text{Prob. 2.38}}$ $\Psi(x,0) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right), & \text{for } 0 \leq x \leq a \\ 0, & \text{else} \end{cases}$

(a) $P_{E_n} = |c_n|^2$

$$c_n = \int_0^{2a} \psi_{n,2a}^*(x)\, \Psi(x,0)\, dx$$

$$= \sqrt{\frac{2}{2a}} \int_0^a \sin\left(\frac{n\pi}{2a}x\right) \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) dx$$

$$= \frac{\sqrt{2}}{a} \int_0^a \sin\left(\frac{n\pi}{2a}x\right) \sin\left(\frac{\pi}{a}x\right) dx$$

$$= \frac{1}{\sqrt{2}\,a} \int_0^a \left[\cos\left(\frac{(n-2)}{2a}\pi x\right) - \cos\left(\frac{n+2}{2a}\pi x\right)\right] dx$$

For $n=2$

$$c_2 = \frac{1}{\sqrt{2}\,a} \int_0^a \left(1 - \cos\left(\frac{2\pi}{a}x\right)\right) dx$$

$$= \frac{1}{\sqrt{2}}$$

For $n \neq 2$

$$c_n = \frac{1}{\sqrt{2}\,a} \left[\frac{2a}{(n-2)\pi} \sin\left(\frac{(n-2)\pi}{2a}x\right) - \frac{2a}{(n+2)\pi} \sin(\ )\right]_0^a$$

$$= \begin{cases} 0 & \text{for } n \text{ even} \\ \frac{\sqrt{2}}{\pi}\left(\frac{1}{n-2}\sin\left(\frac{n\pi}{2} - \pi\right) - \frac{1}{n+2}\sin\left(\frac{n\pi}{2} + \pi\right)\right) & n \text{ odd} \end{cases}$$

For "n" odd;

$$= \frac{\sqrt{2}}{\pi}\left(-\frac{1}{n-2} + \frac{1}{n+2}\right) \sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{\sqrt{2}}{\pi}\left(\frac{-n-2+n-2}{n^2-4}\right)(-1)^{\frac{n-1}{2}}$$

$$= \frac{\sqrt{2}}{\pi}\,\frac{4}{n^2-4}\,(-1)^{\frac{n+1}{2}}$$

So $P_{E_n} = \begin{cases} \frac{1}{2}, & n=2 \\ 0, & n = \text{even (not 2)} \\ \frac{32}{\pi^2(n^2-4)^2}, & n = \text{odd} \end{cases}$

$$P_{E_1} = \frac{32}{\pi^2 9} \simeq 0.360$$

So $\sum_{n=3}^{\infty} P_{E_n} = 1 - P_{E_1} - P_{E_2} = 1 - 0.5 - 0.36$
$= 0.14$

So the most probable energy is E_2

$$P_{E_2} = \frac{1}{2}$$

(b) The next probable energy is E_1

$$P_{E_1} = 0.36$$

(c) $\langle H \rangle = \int_0^{2a} \Psi^*(x,t) H \Psi(x,t)\,dx$

$= \int_0^{2a} \Psi^*(x,0+) H \Psi(x,0+)\,dx$

, where $t=0+$ implies the moment right after the well expansion

Because

for $0 < x < a$

$$\Psi^*(x,0+) = \Psi(x,0+) = \begin{cases} \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) \\ 0, & \text{else} \end{cases}$$

$$\langle H \rangle = \int_0^a \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) H \sqrt{\frac{2}{a}} \sin\left(\frac{\pi}{a}x\right) dx$$

$= E_1$ of the well of width "a"

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2$$