HW 2

Note Title 9/23/2010

$\boxed{\text{Prob. 1.7}}$

$$\frac{d\langle p\rangle}{dt} = \frac{\hbar}{i}\int \frac{\partial}{\partial t}\left(\Psi^*\frac{\partial \Psi}{\partial x}\right)dx$$

$$= \frac{\hbar}{i}\int\left[\frac{\partial \Psi^*}{\partial t}\cdot\frac{\partial \Psi}{\partial x} + \Psi^*\frac{\partial}{\partial x}\left(\frac{\partial \Psi}{\partial t}\right)\right]dx$$

$$= \frac{\hbar}{i}\left\{\int\left[\frac{\partial \Psi^*}{\partial t}\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial t}\right]dx + \cancelto{0}{\Psi^*\frac{\partial \Psi}{\partial t}\Big|_{-\infty}^{\infty}}\right\}$$

↑ Integration by parts

$$= \frac{\hbar}{i}\int\left\{\left[-\frac{i\hbar}{2m}\frac{\partial^2 \Psi^*}{\partial x^2} + \frac{i}{\hbar}V\Psi^*\right]\frac{\partial \Psi}{\partial x} - \frac{\partial \Psi^*}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]\right\}dx \quad - (*)$$

↑ Eq 1.23 & 1.24

Let's use integration by parts for the left terms

$$\int \frac{\partial^2 \Psi^*}{\partial x^2}\frac{\partial \Psi}{\partial x}dx = \cancelto{0}{\frac{\partial \Psi^*}{\partial x}\frac{\partial \Psi}{\partial x}\Big|_{-\infty}^{\infty}} - \int\frac{\partial \Psi^*}{\partial x}\frac{\partial^2 \Psi}{\partial x^2}dx$$

$$= -\int\frac{\partial \Psi^*}{\partial x}\frac{\partial^2 \Psi}{\partial x^2}dx$$

$$\int V\Psi^*\frac{\partial \Psi}{\partial x}dx = \cancelto{0}{V\Psi^*\Psi\Big|_{-\infty}^{\infty}} - \int\frac{\partial}{\partial x}(V\Psi^*)\Psi dx$$

$$= -\int\left[\frac{\partial V}{\partial x}\Psi^*\Psi + V\frac{\partial \Psi^*}{\partial x}\Psi\right]dx$$

$$= \left\langle -\frac{\partial V}{\partial x}\right\rangle - \int\left(\frac{\partial \Psi^*}{\partial x}V\Psi\right)dx$$

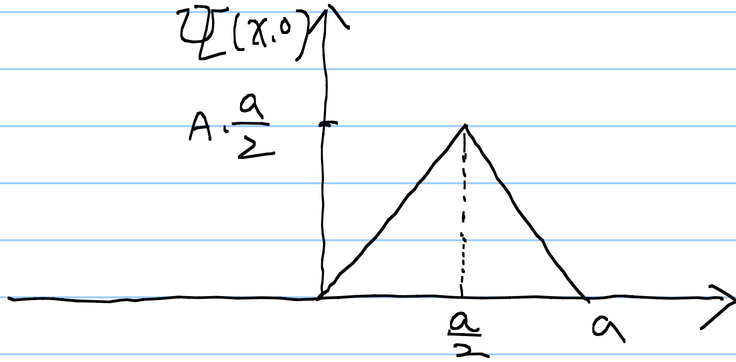
Now

$$(*) \Rightarrow \frac{\hbar}{i}\left\{\int\left[\frac{i\hbar}{2m}\left(\frac{\partial \Psi^*}{\partial x}\frac{\partial^2 \Psi}{\partial x^2}\right) - \frac{i}{\hbar}\frac{\partial \Psi^*}{\partial x}V\Psi\right]dx - \int\frac{\partial \Psi^*}{\partial x}\left[\frac{i\hbar}{2m}\frac{\partial^2 \Psi}{\partial x^2} - \frac{i}{\hbar}V\Psi\right]dx + \frac{i}{\hbar}\left\langle -\frac{\partial V}{\partial x}\right\rangle\right\}$$

$$= \underline{\left\langle \frac{-\partial V}{\partial x}\right\rangle} \qquad \text{Q.E.D.}$$

Prob. 2.7

(a) $$\Psi(x,0) = \begin{cases} Ax, & 0 \le x \le \frac{a}{2} \\ A(a-x), & \frac{a}{2} \le x \le a \end{cases}$$



$$1 = \int_0^a |\Psi(x,0)|^2 dx = 2\int_0^{\frac{a}{2}} A^2 x^2 dx$$

$$= 2A^2 \cdot \frac{1}{3}\left(\frac{a}{2}\right)^3 = A^2 \cdot \frac{1}{3} \cdot \frac{a^3}{4} \Rightarrow A = \left(\frac{12}{a^3}\right)^{\frac{1}{2}}$$

(b) $$c_n = \int_0^a \psi_n(x)\, \Psi(x,0)\, dx$$

$$= \sqrt{\frac{2}{a}} \int_0^a \sin\left(\frac{n\pi}{a}x\right) \Psi(x,0)\, dx$$

$$= \sqrt{\frac{2}{a}} \cdot A \Big[\int_0^{\frac{a}{2}} \sin\left(\frac{n\pi}{a}x\right) x\, dx$$

$$+ \int_{\frac{a}{2}}^{a} \sin\left(\frac{n\pi}{a}x\right)(a-x)\, dx \Big]$$

$$= \sqrt{\frac{2}{a}} \cdot A \Big[-x\left(\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right)\Big|_0^{\frac{a}{2}} + \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{a}x\right)\Big|_0^{\frac{a}{2}}$$

$$- a \cdot \frac{a}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\Big|_{\frac{a}{2}}^{a}$$

$$+ x\left(\frac{a}{n\pi}\cos\left(\frac{n\pi}{a}x\right)\right)\Big|_{\frac{a}{2}}^{a} - \left(\frac{a}{n\pi}\right)^2 \sin\left(\frac{n\pi}{a}x\right)\Big|_{\frac{a}{2}}^{a} \Big]$$

$$= \sqrt{\frac{2}{a}} \cdot A \cdot \frac{a}{n\pi} \Big[-\cancel{\frac{a}{2}\cos\left(\frac{n\pi}{2}\right)} + \frac{a}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$-\cancel{a\cos(n\pi)} + \cancel{a\cos\left(\frac{n\pi}{2}\right)}$$

$$+\cancel{a\cos(n\pi)} - \cancel{\frac{a}{2}\cos\left(\frac{n\pi}{2}\right)}$$

$$- \frac{a}{n\pi}\cancelto{0}{\sin(n\pi)} + \frac{a}{n\pi}\sin\left(\frac{n\pi}{2}\right) \Big]$$

$$= \sqrt{\frac{2}{a}} \cdot \sqrt{\frac{12}{a^3}} \cdot \frac{a}{n\pi} \cdot \frac{2a}{n\pi} \sin\left(\frac{n\pi}{2}\right)$$

$$= \frac{2\sqrt{6}}{a^2} \cdot \frac{2a^2}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right) = \frac{4\sqrt{6}}{(n\pi)^2} \sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} 0, & \text{for } n \text{ even} \\ \frac{4\sqrt{6}}{(n\pi)^2} \cdot (-1)^{\frac{n-1}{2}}, & \text{for } n \text{ odd} \end{cases}$$

So $\Psi(x,t) = \sum_{n=1}^{\infty} c_n \psi_n(x) e^{-i\frac{E_n}{\hbar}t}$

$$= \sum_{n=1,3,5,\ldots}^{\infty} \frac{4\sqrt{6}}{(n\pi)^2} (-1)^{\frac{n-1}{2}} \cdot \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right) \times e^{-i\frac{E_n}{\hbar}t}$$

$$= \frac{8\sqrt{3}}{\sqrt{a}} \cdot \frac{1}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^2} \sin\left(\frac{n\pi}{a}x\right) \times e^{-i\frac{E_n}{\hbar}t}$$

with $E_n = \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2$

(c) $P_{E_1} = |c_1|^2 = \left(\frac{4\sqrt{6}}{\pi^2}\right)^2 = \frac{96}{\pi^4}$

(d) $\langle H \rangle = \sum_n |c_n|^2 E_n$

$$= \sum_{n=1,3,5,\ldots}^{\infty} \frac{96}{\pi^4} \frac{1}{n^4} \cdot \frac{\hbar^2}{2m} \frac{n^2\pi^2}{a^2}$$

$$= \frac{48}{\pi^2} \cdot \frac{\hbar^2}{ma^2} \sum_{n=1,3,5}^{\infty} \frac{1}{n^2} \quad \frac{\pi^2}{8}$$

$$= \frac{6\hbar^2}{ma^2}$$

$\boxed{\text{Prob 2.23}}$

(a) $$\int_{-3}^{+1}(x^3-3x^2+2x-1)\,\delta(x+2)\,dx$$

$$= (-2)^3 - 3(-2)^2 + 2(-2) - 1$$

$$= -8 - 12 - 4 - 1 = -25$$

(b) $$\int_{0}^{\infty}[\cos(3x)+2]\,\delta(x-\pi)\,dx$$

$$= \cos(3\pi) + 2 = -1 + 2 = 1$$

(c) $$\int_{-1}^{1} \exp(|x|+3)\,\delta(x-2)\,dx = 0$$

, because $x=2$ is outside the integration range.

$\boxed{\text{Prob. 2.24}}$

(a) $$\int_{-\infty}^{\infty}\delta(cx)f(x)\,dx = \int_{c\cdot(-\infty)}^{c\cdot\infty}\delta(x')f\left(\frac{x'}{c}\right)\frac{dx'}{c}$$

$$\left[\begin{array}{l} cx = x' \\ \Rightarrow c\,dx = dx' \end{array}\right]$$

$$= \begin{cases} \displaystyle\int_{-\infty}^{\infty}\delta(x')f\left(\frac{x'}{c}\right)\frac{dx'}{c} = \frac{f(0)}{c} & \text{for } c>0 \\[2ex] \displaystyle\int_{\infty}^{-\infty}\delta(x')f\left(\frac{x'}{c}\right)\frac{dx'}{c} = -\frac{f(0)}{c} & \text{for } c<0 \end{cases}$$

$$\therefore \int_{-\infty}^{\infty}\delta(cx)f(x)\,dx = \frac{f(0)}{|c|} = \int_{-\infty}^{\infty}\frac{\delta(x)}{|c|}f(x)\,dx$$

$$\therefore \underline{\delta(cx) = \frac{\delta(x)}{|c|}}$$

(b) $$\int_{-\infty}^{\infty}\frac{d\theta}{dx}f(x)\,dx = \underbrace{\cancel{\theta(x)f(x)\Big|_{-\infty}^{\infty}}}_{\to 0} - \int_{-\infty}^{\infty}\theta(x)\frac{df(x)}{dx}\,dx$$

$$\left[\begin{array}{l} f(\infty) = f(-\infty) \\ = 0 \end{array}\right]$$

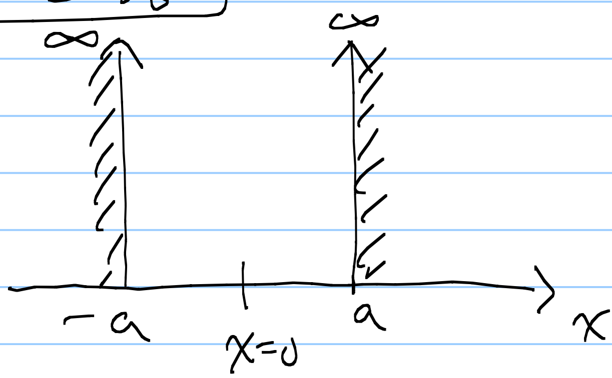
$$= -\int_{0}^{\infty}\frac{df(x)}{dx}\,dx = -f(\infty) + f(0) = f(0)$$

$\uparrow$ $\theta(x) = 0$ for $x<0$ $\qquad\qquad$ $f(\infty) = 0$ $\uparrow$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{d\theta}{dx} f(x)\,dx = f(0) = \int_{-\infty}^{\infty} \delta(x) f(x)\,dx$$

$$\therefore \quad \underline{\frac{d\theta}{dx} = \delta(x)}$$

$\boxed{\text{Prob. 2.36}}$



$V(x) = 0$ for $-a < x < a$

$= \infty$ outside

For $x \in [-a, a]$

$$\psi(x) = A \sin kx + B \cos kx, \quad k = \frac{\sqrt{2mE}}{\hbar}$$

$$\Rightarrow \psi(-a) = -A \sin ka + B \cos ka = 0$$

$$\psi(a) = A \sin ka + B \cos ka = 0$$

$$\Rightarrow 2B \cos ka = 0$$

① If $B = 0$, $\quad A \sin ka = 0 \Rightarrow k = \frac{n\pi}{a}$

$, n = 1, 2, 3, \ldots$

$$\Rightarrow \psi(x) = A \sin kx$$

$$1 = \int_{-a}^{a} |\psi(x)|^2 dx = A^2 \int_{-a}^{a} \sin^2 kx\,dx$$

$$= A^2 \cdot a \quad \Rightarrow A = \frac{1}{\sqrt{a}}$$

$$\therefore \quad \psi(x) = \frac{1}{\sqrt{a}} \sin kx, \quad k = \frac{n\pi}{a}, \quad n = 1, 2 \ldots$$

② If $B \neq 0$, $\cos ka = 0 \Rightarrow k = \frac{(n-\frac{1}{2})\pi}{a}$, $n = 1, 2, 3 \ldots$

$\Rightarrow \quad 2A \sin ka = 0 \Rightarrow A = 0$

because $\sin((n-\frac{1}{2})\pi) \neq 0$

$\Rightarrow \psi_n(x) = B \cos kx$,

$1 = \int_{-a}^{a} |\psi_n(x)|^2 dx = B^2 \int_{-a}^{a} \cos^2(kx) dx$

$= B^2 \cdot a \quad \Rightarrow \quad B = \frac{1}{\sqrt{a}}$

$\therefore \quad \psi_n(x) = \frac{1}{\sqrt{a}} \cos kx, \quad k = \frac{(n-\frac{1}{2})\pi}{a}$,

$$\therefore \quad \psi_n(x) = \begin{cases} \frac{1}{\sqrt{a}} \sin(kx), \text{ for } k = \frac{n\pi}{a} \\ \frac{1}{\sqrt{a}} \cos(kx), \text{ for } k = \frac{(n-\frac{1}{2})\pi}{a} \end{cases}$$

with $E = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \cdot \begin{cases} n^2 \\ (n-\frac{1}{2})^2 \end{cases}$, $n = 1, 2, 3, \ldots$

$$= \frac{\hbar^2 \pi^2}{2ma^2} \cdot \begin{cases} \frac{(2n)^2}{4} \\ \frac{(2n-1)^2}{4} \end{cases}$$

$$= \frac{\hbar^2 \pi^2}{8ma^2} \begin{cases} (2n)^2 \\ (2n-1)^2 \end{cases} = \frac{\hbar^2 \pi^2}{8ma^2} \ell^2$$

, with $\ell = 1, 2, 3, 4, \ldots$

Now from Eq. 2.27, if the width of the wall is replaced by "2a",

$$E_n = \frac{\hbar^2\pi^2}{2m} \cdot \frac{n^2}{(2a)^2} = \frac{\hbar^2\pi^2}{8ma^2} \cdot n^2$$

So this is equivalent to the above result.

From Eq. 2.28 $\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}x\right)$,

if we replace x by $\frac{x+a}{2}$

$$\psi_n(x) = \psi_n\left(\frac{x+a}{2}\right) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a}\frac{x+a}{2}\right)$$

$$= \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{2a}x + \frac{n\pi}{2}\right)$$

$$= \sqrt{\frac{2}{a}} \times \begin{cases} (-1)^{\frac{n}{2}} \cdot \sin\left(\frac{n\pi}{2a}x\right) & : n \text{ even} \\ (-1)^{\frac{n-1}{2}} \cos\left(\frac{n\pi}{2a}x\right) & : n \text{ odd} \end{cases}$$

These are identical to the ones obtained above except for the normalization constants and signs in front.

If we neglect the signs in front, the lowest three states look like