HW 1 solution

Note Title 9/16/2010

1) Prob 1.3 from Griffiths

$$\rho(x) = A e^{-\lambda(x-a)^2}$$

(a) $$1 = A\int_{-\infty}^{\infty} e^{-\lambda(x-a)^2}\,dx$$

$$= A\int_{-\infty}^{\infty} e^{-\lambda u^2}\,du$$

$$= A\sqrt{\frac{\pi}{\lambda}} \Rightarrow \boxed{A = \sqrt{\frac{\lambda}{\pi}}}$$

(b) $$\langle x\rangle = \int_{-\infty}^{\infty} x\,\rho(x)\,dx$$

$$= A\int_{-\infty}^{\infty} x\, e^{-\lambda(x-a)^2}\,dx$$

$$= A\int_{-\infty}^{\infty} (u+a)\, e^{-\lambda u^2}\,du$$

$x-a=u$

$$= A\left[\cancelto{0}{\int_{-\infty}^{\infty} u\, e^{-\lambda u^2}\,du} + a\int_{-\infty}^{\infty} e^{-\lambda u^2}\,du\right]$$

$$= \underline{a}$$

(c) $$\langle x^2\rangle = A\int_{-\infty}^{\infty} x^2 e^{-\lambda(x-a)^2}\,dx$$

$$= A\int_{-\infty}^{\infty} (u+a)^2 e^{-\lambda u^2}\,du$$

$$= A\Big[\int u^2 e^{-\lambda u^2}\,du + \cancelto{0}{2a\int u\, e^{-\lambda u^2}\,du}$$

$$+ a^2\int e^{-\lambda u^2}\,du\Big]$$

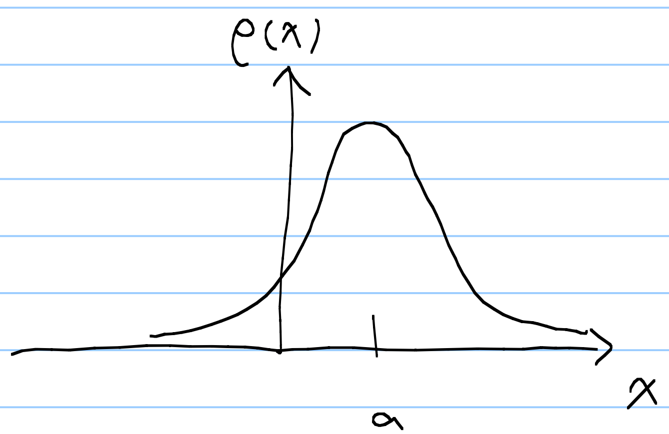
$$= A\left[\frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}} + 0 + a^2\sqrt{\frac{\pi}{\lambda}}\right]$$

$$= a^2 + \frac{1}{2\lambda}$$

$$\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{a^2 + \frac{1}{2\lambda} - a^2}$$

$$= \underbrace{\frac{1}{\sqrt{2\lambda}}}$$

(c)



2) Prob 1.5 from Griffiths

$$\Psi(x,t) = A e^{-\lambda|x|} e^{-i\omega t}$$

(a) $$1 = \int_{-\infty}^{\infty} |\Psi|^2 dx = 2A^2\int_0^{\infty} e^{-2\lambda x} dx$$

$$= \cancel{2}A^2 \cdot \frac{1}{\cancel{2}\lambda} = \frac{A^2}{\lambda} \Rightarrow \underbrace{A = \sqrt{\lambda}}$$

(b) $$\langle x\rangle = \int_{-\infty}^{\infty} \Psi^* x \Psi\, dx = \lambda\int_{-\infty}^{\infty} \underbrace{x e^{-2\lambda|x|}}_{\text{odd integrand}} dx$$

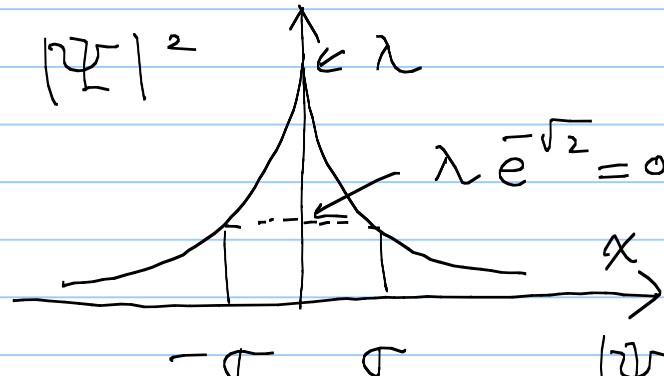
$$= \underbrace{0}$$

$$\langle x^2\rangle = \lambda\int_{-\infty}^{\infty} x^2 e^{-2\lambda|x|} dx$$

$$= 2\lambda\int_0^{\infty} x^2 e^{-2\lambda x} dx = 2\lambda \cdot 2! \left(\frac{1}{2\lambda}\right)^3$$

$$= \frac{1}{2\lambda^2} \;//$$

(c) $\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \underbrace{\frac{1}{\sqrt{2}\,\lambda}}$



$\lambda e^{-\sqrt{2}} = 0.243\lambda$

$|\Psi(\sigma,t)|^2$

$= \lambda e^{-2\lambda \frac{1}{\sqrt{2}\lambda}} = \lambda e^{-\sqrt{2}}$

$$P_{|x|>\sigma} = 2 \int_\sigma^\infty dx\, |\Psi(x,t)|^2$$

$$= 2\lambda \int_{\frac{1}{\sqrt{2}\lambda}}^\infty e^{-2\lambda x}\, dx = e^{-2\lambda \cdot \frac{1}{\sqrt{2}\lambda}}$$

$$= e^{-\sqrt{2}} = \underbrace{0.243}$$

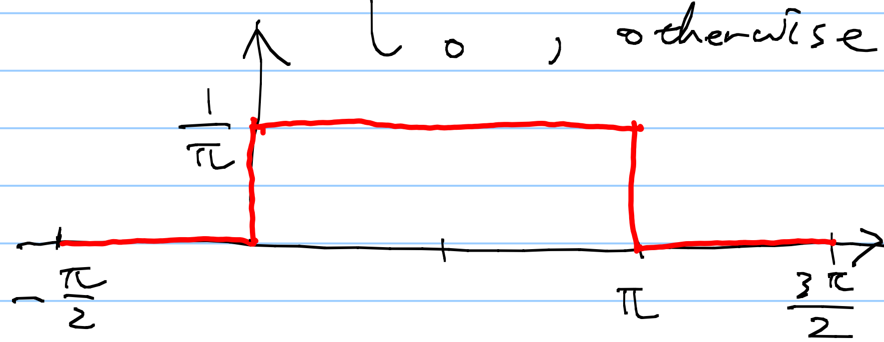
3) Prob. 1.11 from Griffiths

(a) $\rho(\theta) = \begin{cases} \text{const}, & \text{for } 0<\theta<\pi \\ \text{zero}, & \text{otherwise} \end{cases}$

Normalization requires

$$1 = \int_0^\pi \rho(\theta)\, d\theta = \rho_{\text{const}} \cdot \pi$$

$$\Rightarrow \rho(\theta) = \begin{cases} \frac{1}{\pi}, & \text{for } 0<\theta<\pi \\ 0, & \text{otherwise} \end{cases}$$



(b) $\langle \theta \rangle = \int_0^{\pi} \theta \frac{1}{\pi} d\theta = \frac{1}{\pi} \frac{\pi^2}{2} = \frac{\pi}{2}$

$\langle \theta^2 \rangle = \int_0^{\pi} \theta^2 \frac{1}{\pi} d\theta = \frac{1}{\pi} \frac{\pi^3}{3} = \frac{\pi^2}{3}$

$\sigma = \sqrt{\langle \theta^2 \rangle - \langle \theta \rangle^2} = \sqrt{\frac{\pi^2}{3} - \frac{\pi^2}{4}} = \frac{\pi}{2\sqrt{3}}$

(c) $\langle \sin\theta \rangle = \frac{1}{\pi} \int_0^{\pi} \sin\theta \, d\theta = \frac{1}{\pi} (-\cos\theta) \Big|_0^{\pi}$

$= \frac{2}{\pi}$

$\langle \cos\theta \rangle = \frac{1}{\pi} \int_0^{\pi} \cos\theta \, d\theta = \frac{1}{\pi} \sin\theta \Big|_0^{\pi} = 0$

$\langle \cos^2\theta \rangle = \frac{1}{\pi} \int_0^{\pi} \cos^2\theta \, d\theta = \frac{1}{\pi} \int_0^{\pi} \frac{\cos 2\theta + 1}{2} d\theta$

$= \frac{1}{2\pi} \left[\frac{\sin 2\theta}{2} + \theta \right]_0^{\pi} = \frac{1}{2}$

4) $\Psi(x,0) = \begin{cases} A & \text{for } x \in [-1,1] \\ A|x|^{\nu} & \text{otherwise} \end{cases}$

For this state to be physically realizable,

$\int_{-\infty}^{\infty} |\Psi(x,0)|^2 dx < \infty$

$\Rightarrow A^2 \int_1^{\infty} x^{2\nu} dx < \infty$

$\Rightarrow$ if $2\nu+1 \neq 0$, $\frac{x^{2\nu+1}}{2\nu+1} \Big|_1^{\infty} < \infty$

$\Rightarrow \frac{\infty^{2\nu+1}}{2\nu+1} < \infty$

$\Rightarrow 2\nu+1 < 0 \Rightarrow \nu < -\frac{1}{2}$

if $2\nu = -1$, $\ln x \Big|_1^{\infty} = \infty$

So only $\boxed{\nu < -\frac{1}{2}}$ is physically realizable.