Note Title

HW 12 solution

Prob. 8.1.
$$\int_0^a k(x)\,dx = n\pi$$

$$, \quad k(x) = \frac{\sqrt{2m(E - V(x))}}{\hbar}$$

$$\Rightarrow \frac{\sqrt{2m}}{\hbar}\cdot\left[\int_0^{\frac{a}{2}} \sqrt{(E-V_0)}\,dx + \int_{\frac{a}{2}}^{a} \sqrt{E}\,dx\right]$$

$$= \frac{\sqrt{2m}}{\hbar}\left[\sqrt{E-V_0} + \sqrt{E}\right]\frac{a}{2} = n\pi$$

$$\Rightarrow \sqrt{E-V_0} = -\sqrt{E} + \frac{\hbar}{\sqrt{2m}}\,\frac{n\pi}{a}\cdot 2 = -\sqrt{E} + 2\sqrt{E_n^0}$$

$$\Rightarrow \cancel{E} - V_0 = \cancel{E} - 4\sqrt{E}\sqrt{E_n^0} + 4E_n^0$$

$$\Rightarrow 4\sqrt{E}\sqrt{E_n^0} = V_0 + 4E_n^0$$

$$\Rightarrow E = \left[\frac{V_0 + 4E_n^0}{4\sqrt{E_n^0}}\right]^2 = \left[\sqrt{E_n^0} + \frac{V_0}{4\sqrt{E_n^0}}\right]^2$$

$$= E_n^0 + \frac{V_0}{2} + \frac{V_0^2}{16E_n^0}$$

The 1st order perturbation theory in Ex. 6.1 gives
$$E \simeq E_n^0 + \frac{V_0}{2}$$

So if $V_0 \ll E_n^0$ because either V_0 is very small or because n is very large, then the second term above becomes negligible and the two results become equivalent.

2. (a) $P_{0\to 1}(\infty) = |C_{0\to 1}(\infty)|^2 = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} dt\, H'_{10}(t) e^{i\omega_{10}t}\right|^2$

according to the 1st order time-dependent perturbation theory.

Here $\omega_{10} = \frac{E_1 - E_0}{\hbar} = \frac{\frac{3}{2}\hbar\omega - \frac{1}{2}\hbar\omega}{\hbar} = \omega$

for the 1D harmonic oscillator.

$$H'_{10} = \langle 1 | H' | 0 \rangle = \langle 1 | e\vec{E}\cdot\vec{r} | 0 \rangle$$
$$= eE_0 e^{-t^2/T^2} \cdot \langle 1 | x | 0 \rangle$$

Because $x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)$

for the 1D harmonic oscillator,

$$\langle 1 | x | 0 \rangle = \sqrt{\frac{\hbar}{2m\omega}} \langle 1 | a_+ + a_- | 0 \rangle$$
$$= \sqrt{\frac{\hbar}{2m\omega}} \left[\langle 1 | a_+ | 0 \rangle + \langle 1 | \overset{0}{a_- | 0 \rangle} \right]$$
$$= \sqrt{\frac{\hbar}{2m\omega}} \langle 1 | 1 \rangle = \sqrt{\frac{\hbar}{2m\omega}}$$

Thus $P_{0\to 1} = \frac{1}{\hbar^2} \frac{\hbar}{2m\omega} \cdot (eE_0)^2 \left[\int_{-\infty}^{\infty} dt\, e^{-t^2/T^2 + i\omega t}\right]^2$

Now $-\left(\frac{t}{T}\right)^2 + i\omega t = -\frac{1}{T^2}(t^2 - i\omega t T^2)$

$$= -\frac{1}{T^2}\left[\left(t - \frac{i\omega T^2}{2}\right)^2 + \frac{\omega^2 T^4}{4}\right]$$

$$= -\frac{\left(t - \frac{i\omega T^2}{2}\right)^2}{T^2} - \frac{\omega^2 T^2}{4}$$

So $$\int_{-\infty}^{\infty} dt\, e^{-\left(\frac{t}{T}\right)^2 + i\omega t} = \int_{-\infty}^{\infty} dt\, e^{-\frac{\left(t - \frac{i\omega T^2}{2}\right)^2}{T^2}} \cdot e^{-\frac{\omega^2 T^2}{4}}$$

$$= \int_{-\infty}^{\infty} dt'\, e^{-\frac{t'^2}{T^2}} \cdot e^{-\frac{\omega^2 T^2}{4}} = \sqrt{\pi}\, T e^{-\frac{\omega^2 T^2}{4}}$$

$\uparrow$

$$t' \equiv t - \frac{i\omega T^2}{2}$$

$$\therefore P_{0\to 1} = \frac{\pi e^2 E_0^2}{2m\hbar\omega} \cdot T^2 e^{-\frac{\omega^2 T^2}{2}}$$

(b) Because $\langle 2|x|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\langle 2|1\rangle = 0$,

$$P_{0\to 2} = 0$$