HW 10 solution

Prob. 5.5

(a) $H = \begin{cases} \dfrac{p_1^2 + p_2^2}{2m} & \text{for } 0 < x_1, x_2 < a \\ \infty & \text{else} \end{cases}$

$$\psi(x_1, x_2) = \frac{\sqrt{2}}{a}\Big[\sin(\pi x_1/a)\sin(2\pi x_2/a) - \sin(2\pi x_1/a)\sin(\pi x_2/a)\Big]$$

for $0 < x_1, x_2 < a$.

$$H\psi(x_1, x_2) = -\frac{\hbar^2}{2m}\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right]\psi(x_1, x_2)$$

$$= -\frac{\hbar^2}{2m}\cdot\frac{\sqrt{2}}{a}\Big[-\left(\frac{\pi}{a}\right)^2 \sin(\pi x_1/a)\sin(2\pi x_2/a) + \left(\frac{2\pi}{a}\right)^2 \sin(2\pi x_1/a)\sin(\pi x_2/a)$$

$$-\left(\frac{2\pi}{a}\right)^2 \sin(\pi x_1/a)\sin(2\pi x_2/a) + \left(\frac{\pi}{a}\right)^2 \sin(2\pi x_1/a)\sin(\pi x_2/a)\Big]$$

$$= \frac{\hbar^2}{2m}\frac{\sqrt{2}}{a}\left[\left(\frac{\pi}{a}\right)^2 + \left(\frac{2\pi}{a}\right)^2\right]\Big[\sin(\pi x_1/a)\sin\left(\frac{2\pi x_2}{a}\right) - \sin(2\pi x_1/a)\sin(\pi x_2/a)\Big]$$

$$= \frac{\hbar^2}{2m}\left(\frac{\pi}{a}\right)^2 \cdot 5 \cdot \psi(x_1, x_2)$$

$$= 5K\,\psi(x_1, x_2)$$

Thus $\psi(x_1, x_2)$ is the eigenfunction of the Hamiltonian with the eigenvalue $5K$.

(b) $\boxed{\text{distinguishable}}$

2nd excited state

$$\psi_{22} = \frac{2}{a} \sin\left(\frac{2\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right), \quad E_{22} = 8K$$

3rd excited state

$$\psi_{13} = \frac{2}{a} \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{3\pi x_2}{a}\right), \quad E_{13} = 10K$$

$$\psi_{31} = \frac{2}{a} \sin\left(\frac{3\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right), \quad E_{31} = 10K$$

$\boxed{\text{Identical boson}}$

2nd excited state

$$\psi_{22} = \frac{2}{a} \sin\left(\frac{2\pi x_1}{a}\right) \sin\left(\frac{2\pi x_2}{a}\right), \quad E_{22} = 8K$$

3rd excited state

$$\psi_{13} = \frac{\sqrt{2}}{a} \left[\sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{3\pi x_2}{a}\right) + \sin\left(\frac{3\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$, \quad E_{13} = 10K$$

$\boxed{\text{Identical fermion}}$

1st excited state

$$\psi_{13} = \frac{\sqrt{2}}{a} \left[\sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{3\pi x_2}{a}\right) - \sin\left(\frac{3\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right)\right]$$

$$, \quad E_{13} = 10K$$

2nd excited state

$$\psi_{23} = \frac{\sqrt{2}}{a}\left[\sin\left(\frac{2\pi x_1}{a}\right)\sin\left(\frac{3\pi x_2}{a}\right) - \sin\left(\frac{3\pi x_1}{a}\right)\sin\left(\frac{2\pi x_2}{a}\right)\right]$$

$$E_{23} = 13K$$

Prob. 5.7.

(a) distinguishable

$$\psi(x_1, x_2, x_3) = \psi_a(x_1)\,\psi_b(x_2)\,\psi_c(x_3)$$

(b) identical bosons

There are six ways ~~to~~ order the states as shown

x_1	x_2	x_3
a	b	c
	c	b
b	a	c
	c	a
c	a	b
	b	a

Let's use a simplified notation such that

$$\psi_a(x_1)\,\psi_b(x_2)\,\psi_c(x_3) \equiv \psi_{abc}$$

Then

$$\boxed{\psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}\left(\psi_{abc} + \psi_{acb} + \psi_{bac} + \psi_{bca} + \psi_{cab} + \psi_{cba}\right)}$$

, where the normalization constant is simply determined from $\sqrt{\#\text{ of terms}}$

(c) Identical fermions

Similar as in (b), but this time sign should be changed for every interchange of the states. If we start with "+" sign for ψ_{abc}, then any odd # of label changes from abc should have a "−" sign and even # of change should have a "+" sign, as shown:

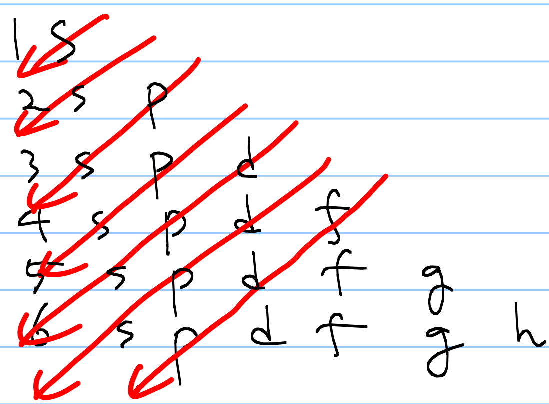
x_1	x_2	x_3	sign
a	b	c	+
	c	b	−
b	a	c	−
	c	a	+
c	a	b	+
	b	a	−

Thus

$$\psi(x_1, x_2, x_3) = \frac{1}{\sqrt{6}}\left(\psi_{abc} - \psi_{acb} - \psi_{bac} + \psi_{bca} + \psi_{cab} - \psi_{cba}\right)$$

3.

Tb has 65 electrons



$\underbrace{1s^2\, 2s^2\, 2p^6\, 3s^2\, 3p^6}_{18}\ \underbrace{4s^2\, 3d^{10}\, 4p^6}_{18}\ 5s^2 \rightarrow$

$\underbrace{4d^{10}\, 5p^6}_{18}\ \underbrace{6s^2\, 4f^9}_{11}$

$\Rightarrow\ 18 + 18 + 18 + 11 = 65$

Outer most partially-filled subshell is $4f^9$. Thus using the Hund's rules

$$\begin{array}{cccccccc} & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\ 4f & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \downarrow\uparrow & \downarrow\uparrow \end{array}$$

$S = \frac{1}{2} \times 5 = \frac{5}{2}$

$L = (-3) + (-2) + (-1) + (0) + 1 + 2\times 2 + 3 \times 2$
$= 5$

$J = L + S = 5 + \frac{5}{2} = \frac{15}{2}$

↑ more than half-filled

$\therefore\ {}^{2S+1}L_J = {}^{6}H_{\frac{15}{2}}$

Prob. 5.16 (a) $E_F = \frac{\hbar^2}{2m}(3\rho\pi^2)^{\frac{2}{3}}$

$$\rho = \frac{N}{V}$$

$$(\text{density})\ d = \frac{\text{Mass}}{V} = \frac{N \cdot \frac{M_A}{N_A}}{V}$$

$\lceil M_A$ = atomic mass of copper = 63.5 g
N_A = Avogadro's number = 6.02×10^{23} $\rfloor$

$$\Rightarrow d = \rho \cdot \frac{M_A}{N_A}$$

$$\Rightarrow \rho = \frac{d \cdot N_A}{M_A} = \frac{8.96\ g/cm^3 \cdot 6.02 \times 10^{23}}{63.5 g}$$

$$= 8.49 \times 10^{22}/cm^3 = 8.49 \times 10^{28}/m^3$$

$m = 9.11 \times 10^{-31}$ kg
$\hbar = 1.05 \times 10^{-34}$ J S

$$E_F = \frac{(1.05 \times 10^{-34} JS)^2}{2 \cdot 9.11 \times 10^{-31} kg}(3 \cdot 8.49 \times 10^{28}\ m^{-3} \cdot \pi^2)^{\frac{2}{3}}$$

$$\times \frac{eV}{1.60 \times 10^{-19} J} = 7.0\ eV$$

(b) $E_F = \frac{1}{2} m v^2$

$$\Rightarrow v = \sqrt{\frac{2E_F}{m}} = \sqrt{\frac{2 \cdot 7.0 eV}{0.51 \times 10^6 eV/c^2}} = 5.2 \times 10^{-3} \cdot c$$

$$= 1.6 \times 10^6\ m/s$$

Because $\frac{v}{c} = 5.2 \times 10^{-3}$, the electrons are non-relativistic

(c) $T = \frac{E_F}{k_B} = \frac{7.0\ eV}{8.62 \times 10^{-5} eV/K} = 8.2 \times 10^4 K$

(d) $P = \dfrac{(3\pi^2)^{2/3}\hbar^2}{5m}\rho^{5/3} = \dfrac{(3\pi^2)^{2/3}(1.05\times10^{-34})}{5\cdot 9.11\times10^{-31}}$

$\times\ (8.49\times10^{28})\ N/m^2 = 3.8\times10^{10}\ N/m^2$

P. 6.2 (a) $E_n^0 = (n+\frac{1}{2})\hbar\omega_0$, the unperturbed.

$$E_n = (n+\tfrac{1}{2})\hbar\omega = (n+\tfrac{1}{2})\hbar\sqrt{\frac{(1+\epsilon)k}{m}}$$
$$= \underbrace{(n+\tfrac{1}{2})\hbar\omega_0\sqrt{1+\epsilon}}$$
$$= (n+\tfrac{1}{2})\hbar\omega_0\left(1+\frac{\epsilon}{2}-\frac{1}{8}\epsilon^2\right)$$

expansion up to 2nd order

(b) $$H = \frac{p^2}{2m}+\frac{1}{2}m\omega^2x^2$$
$$= \frac{p^2}{2m}+\frac{1}{2}m\left(\frac{(1+\epsilon)k}{m}\right)x^2$$
$$= \frac{p^2}{2m}+\frac{1}{2}m\omega_0x^2+\frac{1}{2}\epsilon m\omega_0x^2$$
$$= H^0+\underbrace{\epsilon V^0}_{H'},$$

where $V^0 \equiv \frac{1}{2}m\omega_0x^2$ is the unperturbed potential energy.

$$E_n^1 = \langle\psi_n^0|H'\psi_n^0\rangle$$
$$= \epsilon\langle\psi_n^0|V^0|\psi_n^0\rangle$$
$$= \frac{\epsilon}{2}\cdot(n+\tfrac{1}{2})\hbar\omega_0$$

from Example 2.5 of Griffiths,

thus up to the 1st order

$$E_n = E_n^0+E_n^1 = (n+\tfrac{1}{2})\hbar\omega_0\left(1+\frac{\epsilon}{2}\right)$$

$\therefore$ consistent with (a) up to the 1st order