Final 2010 solution

Note Title 12/22/2010

1. Yes, $H = \frac{p^2}{2m} + V(x)$

If $V(x) = 0$, $H = \frac{p^2}{2m} \Rightarrow [H, p] = 0$

Because H & p can commute, they can share the same eigenfunction.

2. $H = \epsilon \begin{pmatrix} 0 & i \\ -i & -b \end{pmatrix}$

$H^\dagger = \epsilon \begin{pmatrix} 0 & -i \\ i & -b^* \end{pmatrix}$

"H" has to be hermitian $\Rightarrow b = b^*$

$\therefore$ b should be real.

3. $\psi_{n\ell m} \Rightarrow \ell = 1$

(a) $\therefore$ $L^2 \Rightarrow \hbar^2 \ell(\ell+1) = 2\hbar^2$

(b) $S = \frac{1}{2}$, $L = 1$

$\Rightarrow J = (L+S) \ldots |L-S|$

$= \frac{3}{2}$ or $\frac{1}{2}$ are possible

But $m_J = m_S + m_L = \frac{1}{2} + 1 = \frac{3}{2}$

Because $J \geq m_J$

$\Rightarrow J$ should be $\frac{3}{2}$ not $\frac{1}{2}$

4. (a) $H = -\gamma B_0 S_z = -\gamma B_0 \cdot \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(b) $S_y \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \lambda \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\Rightarrow \begin{pmatrix} -\lambda & -i\frac{\hbar}{2} \\ i\frac{\hbar}{2} & -\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$

$\Rightarrow \lambda^2 + (i\frac{\hbar}{2})^2 = 0$

$\Rightarrow \lambda^2 - (\frac{\hbar}{2})^2 = 0 \Rightarrow \lambda = \pm\frac{\hbar}{2}$

$\lambda = \frac{\hbar}{2} \Rightarrow \begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$

$\Rightarrow i\alpha = \beta \Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ i\alpha \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda = -\frac{\hbar}{2} \Rightarrow \beta = -i\alpha \Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$

So $(S_z \Rightarrow \frac{\hbar}{2})$ collapses the state to its eigen state, which is $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$

(c) $\chi(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$

$E \begin{pmatrix} 1 \\ 0 \end{pmatrix} = -\gamma B_0 \frac{\hbar}{2}$

$E \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \gamma B_0 \frac{\hbar}{2}$

There fore

$$\chi(t) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix} e^{-iE_{\binom{1}{0}}t/\hbar} + i\begin{pmatrix}0\\1\end{pmatrix} e^{-iE_{\binom{0}{1}}t/\hbar} \right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{\gamma B_0}{2}t} \\ i\, e^{-i\frac{\gamma B_0}{2}t} \end{pmatrix}$$

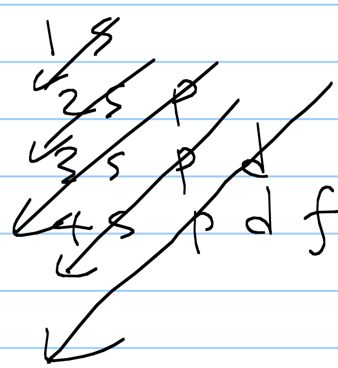
(d) $P(S_z = \frac{\hbar}{2}) = |\langle S_z = \frac{\hbar}{2} | \chi \rangle|^2$

$$= \left[(1 \quad 0) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i\frac{\gamma B_0 t}{2}} \\ i\, e^{-i\frac{\gamma B_0 t}{2}} \end{pmatrix} \right]^2$$

$$= \underline{\frac{1}{2}}$$

$$P(S_z = -\frac{\hbar}{2}) = \underline{\frac{1}{2}}$$

5.



23 electrons

$$(1s)^2 (2s)^2 (2p)^6 (3s)^2 (3p)^6 (4s)^2 (3d)^3$$

	-2	-1	0	1	2
3d			↑	↑	↑

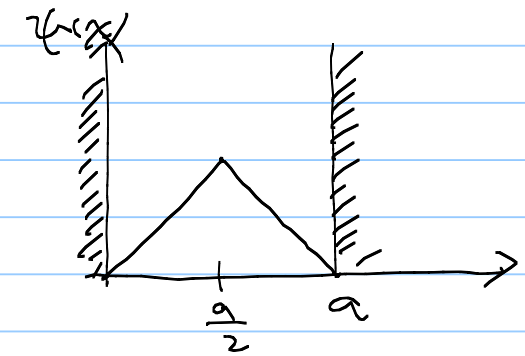
$S = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$

$L = 2 + 1 = 3,$

$J = |L - S| = \frac{3}{2}$ $\therefore$ ${}^4F_{\frac{3}{2}}$

$\boxed{\text{Ex. 2}}$ Find an upper bound on the ground state energy of the 1-D infinite square well, using the triangular trial wave function

$$\psi(x) = \begin{cases} Ax & , \text{ if } 0 \le x \le \frac{a}{2} \\ A(a-x) & , \text{ if } \frac{a}{2} \le x \le a \\ 0 & \text{otherwise} \end{cases}$$



$$1 = |A|^2 \left[\int_0^{\frac{a}{2}} x^2 dx + \int_{\frac{a}{2}}^{a} (a-x)^2 dx \right]$$

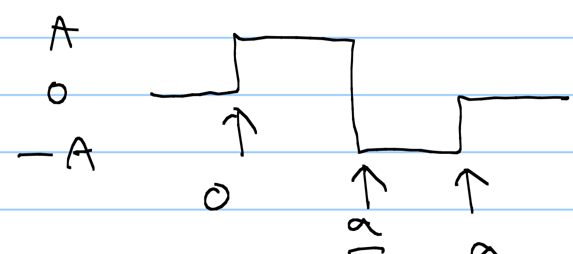
$$= |A|^2 \frac{1}{3} \cdot \left(\frac{a}{2}\right)^3 \cdot 2 = |A|^2 \cdot \frac{a^3}{12}$$

$$\Rightarrow A = \left[\frac{12}{a^3}\right]^{\frac{1}{2}}$$

$$\frac{d\psi}{dx} = \begin{cases} A & , \text{ for } 0 < x < \frac{a}{2} \\ -A & , \text{ for } \frac{a}{2} < x < a \\ 0 & , \text{ else} \end{cases}$$

Using the step functions,

$$\frac{d\psi}{dx} = A\theta(x) - 2A\theta\left(x - \frac{a}{2}\right) + A\theta(x-a)$$



Since $\dfrac{d\theta(x-a)}{dx} = \delta(x-a)$,

$$\frac{d^2\psi}{dx^2} = A\left[\delta(x) - 2\delta(x-\tfrac{a}{2}) + \delta(x-a)\right]$$

Thus

$$\langle H\rangle = \langle T\rangle + \cancel{\langle V\rangle}^{0}$$
$$= -\frac{\hbar^2}{2m}\int \psi(x)\frac{d^2}{dx^2}\psi(x)\,dx$$
$$= -\frac{\hbar^2}{2m}A\int \psi(x)\left[\delta(x) - 2\delta(x-\tfrac{a}{2}) + \delta(x-a)\right]dx$$
$$= -\frac{\hbar^2}{2m}A\left[\cancel{\psi(0)}^{0} - 2\psi(a/2) + \cancel{\psi(a)}^{0}\right]$$
$$= \frac{\hbar^2}{m}A^2\cdot\frac{a}{2} = \frac{\hbar^2}{m}\cdot\frac{a}{2}\cdot\frac{12}{a^3}$$
$$= \underline{\frac{12\hbar^2}{2ma^2}}$$

7. According to the WKB approximation,

$$T \simeq e^{-2\int_0^a \kappa(x)\,dx}$$

with $\kappa(x) = \dfrac{\sqrt{2m(V_0-E)}}{\hbar} = \kappa_0$: const

$$\therefore\ T_0 \simeq e^{-2\int_0^a dx\cdot\kappa_0} = e^{-2\kappa_0 a}$$

$$T_{2a} \simeq e^{-2\int_0^{2a} dx\,\kappa_0} = e^{-4\kappa_0 a}$$
$$= \underline{T_0^{\,2}}$$

8. $H'(t) = -q\vec{E}\cdot\vec{r} = eE(t)y$

Because

$$P_{i\to f}(t) = |C_f(t)|^2$$

$$= \frac{1}{\hbar^2}\left|\int_0^t H'_{fi}(t) e^{i\frac{E_f - E_i}{\hbar}t}\,dt\right|^2$$

If $H'_{fi}(t)$ is zero, then that transition does not occur, i.e. forbidden.

Because $H'_{fi}(t) = eE(t)\langle f|y|i\rangle$

with $|i\rangle \equiv |100\rangle$

$$|f\rangle = \begin{cases} |200\rangle \\ |21\pm1\rangle \\ |210\rangle \end{cases}$$

So $\langle 200|y|100\rangle$

$$= \langle R_{20}|r|R_{10}\rangle\langle Y_0^0|\sin\theta\sin\varphi|Y_0^0\rangle$$

$$\langle Y_0^0|\sin\theta\sin\varphi|Y_0^0\rangle = \frac{1}{4\pi}\int_0^\pi \sin\theta\sin\theta\,d\theta \cdot \int_0^{2\pi}\sin\varphi\,d\varphi \to 0$$

$\therefore$ $|100\rangle \Rightarrow |200\rangle$ is forbidden

For $\langle 210 | y | 100 \rangle$

we need to consider $\langle Y_1^0 | \sin\theta \sin\varphi | Y_0^0 \rangle$

$$\propto \int_0^{2\pi} \sin\varphi \, d\varphi = 0$$

$\therefore$ $|100\rangle \Rightarrow |210\rangle$ is forbidden.

for $\langle 21\pm1 | y | 100 \rangle$

we need to consider $\langle Y_1^{\pm 1} | \sin\theta \sin\varphi | Y_0^0 \rangle$

$$\propto \underbrace{\int_0^{\pi} \sin\theta \sin\theta \, d\theta}_{\text{non-zero}} \int_0^{2\pi} e^{\pm i\varphi} \sin\varphi \, d\varphi$$

$$\int_0^{2\pi} e^{\pm i\varphi} \sin\varphi \, d\varphi$$

$$\propto \int_0^{2\pi} e^{\pm i\varphi} (e^{i\varphi} - e^{-i\varphi}) \, d\varphi$$

$$\underset{\text{for "+"}}{=} \int_0^{2\pi} (e^{2i\varphi} - 1) \, d\varphi$$

$$= \frac{e^{4\pi i} - e^0}{2i} - 2\pi = -2\pi \neq 0$$

similarly "−" sign gives non zero value

$\therefore$ $|100\rangle \Rightarrow |21\pm1\rangle$ transitions are allowed