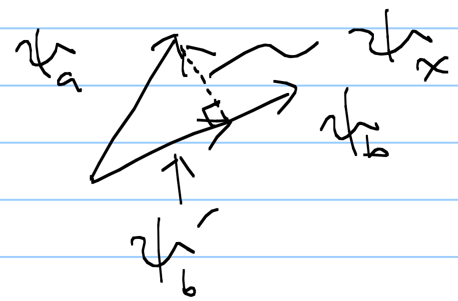
417 final 2009

Note Title 12/18/2009

1. Ans : for any arbitrary state :
   See page 115 of Griffiths
2. Ans : for any stationary state but not for any arbitrary state : See page 26 of Griffiths.
3. Ans : the same as #1, which is for any arbitrary state
   : see Griffiths, page 115.
4. No . Because $H = \frac{p^2}{2m} + V(x)$, $[x, p] = i\hbar$, x and H does not commute . Any non-commuting observables cannot share the same eigen state.

5. $|\psi_a\rangle = |\psi_1\rangle + 2i|\psi_2\rangle = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$

$|\psi_b\rangle = i|\psi_1\rangle + |\psi_2\rangle = \begin{pmatrix} i \\ 1 \end{pmatrix}$



$, \ |\psi_{bn}\rangle = \dfrac{|\psi_b\rangle}{\|\psi_b\|}$

$\psi_x = \psi_a - \psi_b'$

$|\psi_b'\rangle = |\psi_{bn}\rangle \cdot \langle \psi_{bn} | \psi_a \rangle$

$= \dfrac{1}{\|\psi_b\|^2} |\psi_b\rangle\langle\psi_b|\psi_a\rangle$

$\langle\psi_b|\psi_a\rangle = (-i, 1)\begin{pmatrix} 1 \\ 2i \end{pmatrix} = -i + 2i = i$

$\|\psi_b\|^2 = (-i \ \ 1)\begin{pmatrix} i \\ 1 \end{pmatrix} = 2$

$\therefore \ |\psi_b'\rangle = \frac{i}{2}\begin{pmatrix} i \\ 1 \end{pmatrix}$

$$\psi_x = \begin{pmatrix} 1 \\ 2i \end{pmatrix} - \frac{i}{2}\begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2i \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ \frac{i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \frac{1}{2} \\ 2i - \frac{i}{2} \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ \frac{3i}{2} \end{pmatrix} = \frac{3}{2}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$\therefore \ \psi_{x\,norm} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$

check $\langle \psi_{x\,norm} | \psi_b \rangle = (1 \ -i)\begin{pmatrix} i \\ 1 \end{pmatrix}$

$$= i - i = 0$$

$\therefore$ $\psi_{x\,norm}$ is orthogonal to ψ_b

6. Spatial part: $\psi_{321} \Rightarrow n=3, \ l=2, \ m=1$
spin part: $s = \frac{1}{2}, \ m = -\frac{1}{2}$

(a) $L^2 \Rightarrow \hbar^2 l(l+1) = \hbar^2 \cdot 2 \cdot 3 = 6\hbar^2$

(b) $J = |l+s| \ldots |l-s|$

$$\Rightarrow 2 + \frac{1}{2} = \frac{5}{2} \quad \text{or} \quad 2 - \frac{1}{2} = \frac{3}{2}$$

$$m_J = m_l + m_s = 1 - \frac{1}{2} = \frac{1}{2}$$

7. $Y = A\sin\theta\cos\phi = \frac{A}{2}\sin\theta\left(e^{i\phi} + e^{-i\phi}\right)$

Now $Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$

$$Y_1^{-1} = \sqrt{\frac{3}{8\pi}} \sin\theta \, e^{-i\phi}$$

$$\Rightarrow Y = \frac{A}{2} \cdot \left(-\sqrt{\frac{8\pi}{3}} Y_1^1 + \sqrt{\frac{8\pi}{3}} Y_1^{-1} \right)$$

$$= -\sqrt{\frac{2\pi}{3}} A \left(Y_1^1 - Y_1^{-1} \right)$$

Because $\langle Y_1^1 | Y_1^1 \rangle = 1$, $\langle Y_1^{-1} | Y_1^{-1} \rangle = 1$

$$\Rightarrow \langle Y | Y \rangle = 1 \Rightarrow \frac{2\pi}{3} A^2 (1+1) = 1$$

$$\Rightarrow A^2 = \frac{3}{4\pi} \Rightarrow \underbrace{A = \sqrt{\frac{3}{4\pi}}}$$

$$\Rightarrow Y = -\frac{1}{\sqrt{2}} \left(Y_1^1 - Y_1^{-1} \right)$$

(b) $Y_l^m \Rightarrow$ so $l = 1$, 100%

$m = 1$, 50%

$m = -1$, 50%

8. (a) $H = -\alpha B_0 S_z = -\frac{\hbar}{2} \alpha B_0 \sigma_z$

$$= -\frac{\hbar}{2} \alpha B_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) $S_x \Rightarrow \frac{\hbar}{2}$ means the state should be the corresponding eigenstate

$$S_x \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = b$$

$\therefore$ The eigenstate should be $\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

$$\Rightarrow \chi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(c) Because $\chi(0) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

, and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the eigenstate of the Hamiltonian with $E_1 = -\frac{\hbar}{2}\gamma B_0$

and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is that for $E_2 = \frac{\hbar}{2}\gamma B_0$

$$\chi(t) = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\frac{E_1}{\hbar}t} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\frac{E_2}{\hbar}t}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\frac{\gamma B_0}{2}t} \\ e^{-i\frac{\gamma}{2}B_0 t} \end{pmatrix}$$

(d) $\langle S_x \rangle = \langle \chi(t) | S_x | \chi_t \rangle$

$$= \frac{1}{2}\left(e^{-i\frac{\gamma B_0}{2}t} \quad e^{i\frac{\gamma B_0}{2}t} \right) \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

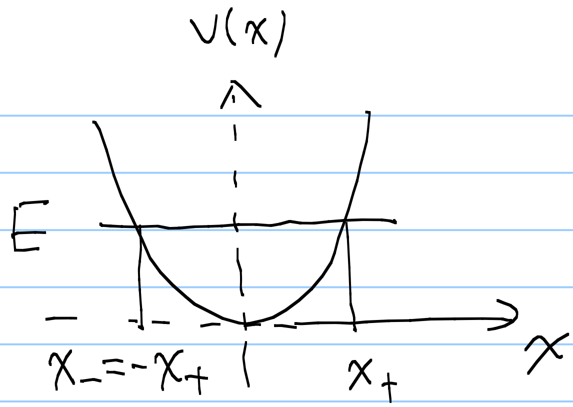
$$\cdot \begin{pmatrix} e^{i\frac{\gamma B_0}{2}t} \\ e^{-i\frac{\gamma}{2}B_0 t} \end{pmatrix}$$

$$= \frac{\hbar}{4} \begin{pmatrix} e^{-i\frac{\gamma B_0 t}{2}} & e^{i\frac{\gamma}{2}B_0 t} \end{pmatrix} \begin{pmatrix} e^{-i\frac{\gamma B_0 t}{2}} \\ e^{\frac{i\gamma B_0 t}{2}} \end{pmatrix}$$

$$= \frac{\hbar}{4} \left(e^{-i\gamma B_0 t} + e^{i\gamma B_0 t} \right)$$

$$= \underline{\frac{\hbar}{2} \cos(\gamma B_0 t)}$$

9.



$$V(x) = \frac{1}{2} m\omega^2 x^2$$

$$E = \frac{1}{2} m\omega^2 x_+^2 \Rightarrow x_+ = \sqrt{\frac{2E}{m\omega^2}}$$

$$\int_{x_-}^{x_+} k(x)\,dx = \left(n-\frac{1}{2}\right)\pi, \quad k(x) = \frac{\sqrt{2m(E-V(x))}}{\hbar}$$

$$\Rightarrow \frac{\sqrt{2mE}}{\hbar}\int_{-x_+}^{x_+}\sqrt{1-\frac{m\omega^2}{2E}x^2}\,dx = \left(n-\frac{1}{2}\right)\pi$$

$$\Rightarrow \qquad \sqrt{\frac{m\omega^2}{2E}}\,x = y \Rightarrow \sqrt{\frac{m\omega^2}{2E}}\,dx = dy$$

$$\Rightarrow \frac{\sqrt{2mE}}{\hbar}\cdot\sqrt{\frac{2E}{m\omega^2}}\int_{-1}^{1}\sqrt{1-y^2}\,dy = \left(n-\frac{1}{2}\right)\pi$$

$$\Rightarrow 2\underbrace{\frac{\sqrt{2mE}}{\hbar}\sqrt{\frac{2E}{m\omega^2}}}_{=\,4\cdot\frac{E}{\hbar\omega}}\int_{0}^{1}\sqrt{1-y^2}\,dy = \left(n-\frac{1}{2}\right)\pi$$

$$y = \sin\theta \Rightarrow 1-y^2 = 1-\sin^2\theta = \cos^2\theta$$
$$dy = \cos\theta\,d\theta$$

$$\Rightarrow \int_{0}^{\frac{\pi}{2}}\cos^2\theta\,d\theta = \frac{\hbar\omega}{4E}\left(n-\frac{1}{2}\right)\pi$$

$$\Rightarrow \int_{0}^{\frac{\pi}{2}}\frac{\cos 2\theta+1}{2}\,d\theta = \frac{1}{2}\left[\frac{\sin 2\theta}{2}+\theta\right]_{0}^{\frac{\pi}{2}}$$

$$\Rightarrow \frac{\cancel{\pi}}{4} = \frac{\hbar\omega}{4E}\left(n-\frac{1}{2}\right)\cancel{\pi}$$

$$\Rightarrow E = \hbar\omega\left(n-\frac{1}{2}\right), \quad n = 1, 2, \dots$$

10. $V(x) = -\alpha\,\delta(x)$

$E_{gs} \leq \langle \psi | H | \psi \rangle \equiv \langle H \rangle$

$H = \frac{p^2}{2m} + V(x) = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \alpha\,\delta(x)$

Let's first normalize the wave function

$\psi(x) = A\, e^{-bx^2}$

$1 = \langle \psi | \psi \rangle = A^2 \int_{-\infty}^{\infty} e^{-2bx^2}\, dx$

$= A^2 \cdot \sqrt{\frac{\pi}{2b}} \Rightarrow A = \left(\frac{2b}{\pi}\right)^{\frac{1}{4}}$

$\langle H \rangle = \langle T \rangle + \langle V \rangle$

$\langle T \rangle = -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \psi^* \frac{d^2}{dx^2}\psi\, dx$

$= -\frac{\hbar^2}{2m} A^2 \int_{-\infty}^{\infty} e^{-bx^2} \frac{d^2}{dx^2} e^{-bx^2}\, dx$

$= +\frac{\hbar^2}{2m} \cdot \sqrt{\frac{2b}{\pi}} \cdot \sqrt{\frac{\pi b}{2}}$

$= \frac{\hbar^2}{2m} b$

$\langle V \rangle = -\alpha A^2 \int_{-\infty}^{\infty} e^{-2bx^2}\, \delta(x)\, dx$

$= -\alpha \sqrt{\frac{2b}{\pi}} \cdot e^0 = -\alpha\sqrt{\frac{2b}{\pi}}$

$$\therefore \langle H \rangle = \frac{\hbar^2}{2m} b - \alpha\sqrt{\frac{2b}{\pi}}$$

$$\Rightarrow \frac{\partial \langle H \rangle}{\partial b} = 0 = \frac{\hbar^2}{2m} - \alpha\sqrt{\frac{2}{\pi}} \cdot \frac{1}{2}\, b^{-\frac{1}{2}}$$

$$= \frac{\hbar^2}{2m} - \alpha\sqrt{\frac{1}{2\pi}}\, b^{-\frac{1}{2}}$$

$$\Rightarrow b^{\frac{1}{2}} = \alpha\sqrt{\frac{1}{2\pi}} \cdot \frac{2m}{\hbar^2} = \alpha\sqrt{\frac{2}{\pi}} \cdot \frac{m}{\hbar^2}$$

$$\Rightarrow b = \frac{m^2\alpha^2}{\hbar^4}\,\frac{2}{\pi}$$

$$\therefore H_{min} = \frac{\hbar^2}{2m} b \left(1 - \alpha\sqrt{\frac{2}{\pi}} \cdot \frac{2m}{\hbar^2} \cdot b^{-\frac{1}{2}} \right)$$

$$\alpha\sqrt{\frac{2}{\pi}}\,\frac{2m}{\hbar^2} \cdot b^{-\frac{1}{2}} = \alpha\sqrt{\frac{2}{\pi}}\,\frac{2m}{\hbar^2} \cdot \alpha^{-1}\sqrt{2\pi} \cdot \frac{\hbar^2}{2m}$$

$$= 2$$

$$\therefore H_{min} = \frac{\hbar^2}{2m} \cdot b\,(1-2)$$

$$= -\frac{\hbar^2}{2m} b = -\frac{\cancel{\hbar^2}}{\cancel{2}m}\left(\frac{m^2\alpha^2}{\cancel{\hbar^4}_{2}}\,\frac{\cancel{2}}{\pi} \right)$$

$$= -\frac{m\alpha^2}{\hbar^2\pi}$$

11. (a) $\psi_{n_x n_y}(x, y) = \frac{2}{a} \sin\left(\frac{n_x \pi}{a} x\right) \sin\left(\frac{n_y \pi}{a} y\right)$

$\Rightarrow$ 1st excited states are $(n_x, n_y) = (1, 2)$ or (2.1)

So $\psi_a = \psi_{1,2} = \underline{\frac{2}{a} \sin\left(\frac{\pi}{a} x\right) \sin\left(\frac{2\pi}{a} y\right)}$

$\psi_b = \psi_{2,1} = \underline{\frac{2}{a} \sin\left(\frac{2\pi}{a} x\right) \sin\left(\frac{\pi}{a} y\right)}$

with $E = \frac{\hbar^2}{2m}\left(\frac{\pi^2}{a^2} + \frac{4\pi^2}{a^2}\right)$

$= \underline{\frac{\hbar^2}{2m} \frac{5\pi^2}{a^2}}$

(b)

$$H'_{aa} = \left(\frac{2}{a}\right)^2 \int_0^a \int_0^a \sin^2\left(\frac{\pi}{a} x\right) \sin^2\left(\frac{2\pi}{a} y\right) \cdot a^2 V_0 \, \delta\left(x - \frac{a}{4}\right) \delta\left(y - \frac{a}{4}\right) dx\,dy$$

$= 4V_0 \sin^2\left(\frac{\pi}{4}\right) \sin^2\left(\frac{\pi}{2}\right)$

$= 2V_0$

$H'_{bb} = 4V_0 \sin^2\left(\frac{\pi}{2}\right) \sin^2\left(\frac{\pi}{4}\right)$

$= 2V_0$

$$H'_{ab} = \left(\frac{2}{a}\right)^2 \iint \sin\left(\frac{\pi}{a} x\right) \sin\left(\frac{2\pi}{a} y\right) \sin\left(\frac{2\pi}{a} x\right) \sin\left(\frac{\pi}{a} y\right) \cdots dx\,dy$$

$$= 4V_0 \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right)$$
$$\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$$
$$= 2V_0$$

$H'_{ba} = 2V_0$

$$\therefore H' = 2V_0 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Let's solve for the eigenvalues,

$$H'\psi = 2V_0 \cdot \lambda \psi$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 1 = 0$$

$$\Rightarrow (\lambda-1-1)(\lambda-1+1) = 0$$

$$\therefore \lambda = 2 \text{ or } 0$$

So the 1st order corrections will be

$4V_0$ and 0

$\therefore$ The new energies are

$$\frac{\hbar^2}{2m}\frac{5\pi^2}{a^2} + 4V_0$$

and $$\frac{\hbar^2}{2m}\frac{5\pi^2}{a^2}$$

12. $H'(t) = -q\vec{E}\cdot\vec{r} = eE(t)\,z$

Because

$$P_{i\to f}(t) = |C_f(t)|^2$$

$$= \frac{1}{\hbar^2}\left|\int_0^t H'_{fi}(t)\, e^{i\frac{E_f - E_i}{\hbar}t}\,dt\right|^2$$

If $H'_{fi}(t)$ is zero, then that transition does not occur, i.e. forbidden.

Because $H'_{fi}(t) = eE(t)\langle f|z|i\rangle$

with $|i\rangle \equiv |100\rangle$

$$|f\rangle = \begin{cases} |200\rangle \\ |21\pm1\rangle \\ |210\rangle \end{cases}$$

So $\langle 200|z|100\rangle$

$$= \langle R_{20}|r|R_{10}\rangle\langle Y_0^0|\cos\theta|Y_0^0\rangle$$

$$\langle Y_1^0|\cos\theta|Y_0^0\rangle = \frac{1}{4\pi}\int_\theta\int_\varphi \cos\theta\sin\theta\,d\theta\,d\varphi$$

$$= -\frac{1}{4\pi}\int_{\theta=0}^{\pi}\cos\theta\,d\cos\theta\cdot 2\pi$$

$$= \frac{1}{2\pi}\int_{-1}^{1} x\,dx = 0 \Rightarrow \text{forbidden}$$

For $\langle 210 | z | 100 \rangle$

We need to consider $\langle Y_1^0 | \cos\theta | Y_0^0 \rangle$

$$\propto \iint_{\theta\,\varphi} \cos^2\theta \sin\theta \, d\theta \, d\varphi$$

$$\propto -\int_{\theta=0}^{\pi} \cos^2\theta \, d\cos\theta = \int_{-1}^{1} x^2 \, dx \quad [x = \cos\theta]$$

$$= \frac{2}{3} \neq 0 \quad \rightarrow \text{allowed}$$

For $\langle 21\pm1 | z | 100 \rangle$

We need to consider

$$\langle Y_1^{\pm1} | \cos\theta | Y_0^0 \rangle$$

$$\propto \iint \sin\theta \, e^{\pm i\phi} \cos\theta \sin\theta \, d\theta \, d\varphi$$

$$= \int_{\theta=0}^{\pi} \sin^2\theta \cos\theta \, d\theta \int_0^{2\pi} e^{\pm i\phi} d\varphi$$

$$\sin\theta = x$$

$$\propto \cancelto{0}{\int_0^0 x^2 dx} \cdot \cancelto{0}{\frac{e^{\pm 2\pi i} - 1}{\pm i}}$$

$$= 0 \quad \Rightarrow \text{forbidden}$$

$\therefore$ transitions from $|100\rangle$ to $|200\rangle$ and $|21\pm1\rangle$ are forbidden.