417 Final Exam, Dec. 18 2009

Student Name: First: Last:

Useful formula

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$

$\cos 2\theta = 2\cos^2\theta - 1$

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1. (2pts) If Hamiltonian does not have an explicit time dependence, the expectation value of the Hamiltonian is constant in time
   (a) for any arbitrary state
   (b) for any stationary state but not for any arbitrary state
   (c) None of the above

2. (2pts) If an observable Q does not commute with Hamiltonian and does not have an explicit time dependence, the expectation value of this observable is constant in time
   (a) For any arbitrary state
   (b) For any stationary state but not for any arbitrary state
   (c) None of the above

3. (2pts) If an observable Q commutes with Hamiltonian and does not have an explicit time dependence, the expectation value of this observable is constant in time
   (a) For any arbitrary state
   (b) For any stationary state but not for any arbitrary state
   (c) None of the above

4. (4pts) Can the position operator "x" and a Hamiltonian share the same eigenfunction? Explain.

5. (8pts) Given $|\psi_a\rangle = |\psi_1\rangle + 2i|\psi_2\rangle$ and $|\psi_b\rangle = i|\psi_1\rangle + |\psi_2\rangle$, where $|\psi_1\rangle$ and $|\psi_2\rangle$ form an orthonormal basis, following the method we discussed in class, construct a state out of $|\psi_a\rangle$ that is orthonormal to $|\psi_b\rangle$, that is your final state should be both orthogonal to $|\psi_b\rangle$ and normalized.

6. An electron with spin down is in the ψ_{321} state of hydrogen atom.
   (a) (2pts) If you measure L^2 of this electron, where L is the orbital angular momentum operator, what value(s) would you get?

   (b) (4pts) What are possible values for the total angular momentum quantum number, J, and the total magnetic quantum number, m, of this electron.

7. Angular wavefunction of a particle is given by $Y(\theta, \varphi) = A\sin\theta\cos\varphi$ (Hint: Express it by a sum of spherical harmonics)
   (a) (4 pts) Find the normalization constant A.

   (b) (4 pts) If you measure the orbital angular momentum quantum numbers l and m of this particle, what values would you get and with what probabilities?

8. A spin-1/2 particle at rest in a uniform magnetic field pointing in the z-direction is described by the Hamiltonian:

$$H = -\gamma B_0 S_z.$$

   (a) (2 pts) Write down the matrix describing this Hamiltonian: our basis is the standard $|m_z = \frac{1}{2}\rangle = \binom{1}{0}$ and $|m_z = -\frac{1}{2}\rangle = \binom{0}{1}$.

   (b) (4 pts) If, at t = 0, measurement of S_x resulted in $\frac{\hbar}{2}$, what is the spinor $\chi(t=0)$ of the state right after the measurement? (Do not forget to normalize.)

   (c) (4 pts) At a later time t (>0), what is the corresponding spinor $\chi(t)$?

   (d) (4 pts) Evaluate $\langle S_x \rangle$ at time t.

9. (8pts) A particle is in a harmonic potential well, that is, $V(x) = \frac{1}{2}m\omega^2 x^2$. Using the WKB approximation, find its energy spectrum. (Note: the final answer will be the same as the standard harmonic oscillator spectrum. If not, check your algebra more carefully.)

10. (8pts) Using the Gaussian function ($\psi(x) = Ae^{-bx^2}$) as a trial wave function with the variational principle, estimate the upper bound for the ground state energy of a delta function potential, $V(x) = -\alpha\delta(x)$. Integrals that may be useful: $\int_{-\infty}^{\infty} e^{-2bx^2}\,dx = \sqrt{\frac{\pi}{2b}}$ and $\int_{-\infty}^{\infty} e^{-bx^2} \frac{d^2}{dx^2}\left(e^{-bx^2}\right)dx = -\sqrt{\frac{\pi b}{2}}$.

11. Suppose we perturb 2D infinite square potential well (V(x,y) = 0 if 0 < x, y < a, V(x,y)= ∞ otherwise) by putting a delta function "bump" at the point (a/4, a/4):

$$H' = a^2 V_0 \delta\left(x - \frac{a}{4}\right)\delta\left(y - \frac{a}{4}\right)$$

   (a) (4pts) Write down the single-particle wave functions and the energy of the doubly degenerate first excited state of the unperturbed 2D infinite square well.

(b) (8pts) Now if you take the perturbation into account, up to the first order, what will be the new energies resulting from this first excited state?

12. (8pts) A hydrogen atom is placed in a time-dependent electric field $\mathbf{E} = E(t)\hat{z}$. According to the first-order time-dependent perturbation theory, this electrical field can cause the electron of the hydrogen atom to make transitions between (otherwise stationary) states. If the electron is initially in its ground state ($|nlm\rangle = |100\rangle$) and if we consider its transition to the quadruply degenerate first excited states ($|nlm\rangle = \{|200\rangle, |21 \pm 1\rangle, |210\rangle\}$), which of these four transitions are forbidden based on the first order time-dependent perturbation theory? Remember that $z = r\cos\theta$ in spherical coordinates.