Final 1

1. (a) $H = -\gamma B_0 S_z$

$$= -\frac{\hbar}{2}\gamma B_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad E_\uparrow = -\frac{\hbar}{2}\gamma B_0, \quad E_\downarrow = \frac{\hbar}{2}\gamma B_0$$

(b) $S_x \chi = \frac{\hbar}{2}\chi$

$$\Rightarrow \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0 \Rightarrow a = b$$

$$\therefore \chi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(c)

$$\chi(t) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\frac{E_\uparrow}{\hbar}t} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-i\frac{E_\downarrow}{\hbar}t}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} e^{+i\frac{\gamma B_0}{2}t} \\ e^{-i\frac{\gamma B_0}{2}t} \end{pmatrix}.$$

(d) $\langle S_y \rangle = \langle \chi(t) | S_y | \chi(t) \rangle$

$$= \frac{1}{2}\begin{pmatrix} e^{-i\frac{\gamma B_0}{2}t} & e^{+i\frac{\gamma B_0}{2}t} \end{pmatrix} \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} e^{i\frac{\gamma B_0 t}{2}} \\ e^{-i\frac{\gamma B_0}{2}t} \end{pmatrix}$$

$$= \frac{\hbar}{4}\begin{pmatrix} e^{-i\frac{\gamma B_0 t}{2}} & e^{i\frac{\gamma B_0 t}{2}} \end{pmatrix}\begin{pmatrix} -i e^{-i\frac{\gamma B_0 t}{2}} \\ i e^{i\frac{\gamma B_0 t}{2}} \end{pmatrix}$$

$$= i\frac{\hbar}{4}\left(-e^{-i\gamma B_0 t} + e^{i\gamma B_0 t}\right) = -\frac{\hbar}{2}\sin\gamma B_0 t$$

2. (a) $H = -\frac{\hbar}{2}\gamma B_0 \cos(\omega t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

(b) $\chi(t=0) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$, $\chi(t) = \begin{pmatrix}a\\b\end{pmatrix}$

$$i\hbar \frac{d}{dt}\chi(t) = H(t)\,\chi(t)$$

$$\Rightarrow i\hbar\frac{d}{dt}\begin{pmatrix}a\\b\end{pmatrix} = -\frac{\hbar}{2}\gamma B_0\cos(\omega t)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$= -\frac{\hbar}{2}\gamma B_0\cos(\omega t)\begin{pmatrix}a\\-b\end{pmatrix}$$

$$\Rightarrow \dot{a} = i\frac{\hbar}{2}\gamma B_0\cos(\omega t)\,a \quad — \; *$$

$$\dot{b} = -i\frac{\hbar}{2}\gamma B_0\cos(\omega t)\,a \quad — \; **$$

$$* \Rightarrow \frac{da}{a} = i\frac{\hbar}{2}\gamma B_0\cos(\omega t)\,dt$$

$$\Rightarrow \ln a = i\frac{\hbar}{2}\gamma B_0\frac{\sin(\omega t)}{\omega} + \text{const}$$

$$\Rightarrow a(t) = a_0\, e^{i\frac{\hbar}{2}\gamma B_0\frac{\sin(\omega t)}{\omega}}$$

$$= \underbrace{\frac{1}{\sqrt{2}}\, e^{i\frac{\hbar\gamma}{2}B_0\frac{\sin(\omega t)}{\omega}}}$$

$\uparrow\; a(t=0) = \frac{1}{\sqrt{2}}$.

$** \Rightarrow$ similarly $\quad \underline{b(t) = \frac{1}{\sqrt{2}}\, e^{-i\frac{\hbar\gamma}{2}B_0\frac{\sin(\omega t)}{\omega}}}$

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So $$\chi(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i \frac{\hbar \gamma}{2} B_0 \frac{\sin(\omega t)}{\omega}} \\ e^{-i \frac{\hbar \gamma}{2} B_0 \frac{\sin(\omega t)}{\omega}} \end{pmatrix}.$$

3. $\psi_{310} \Rightarrow l=1,\ m_l=0$.

spin down $\Rightarrow s=\frac{1}{2},\ m_s=-\frac{1}{2}$

total spin $\Rightarrow j=?,\ m_J=-\frac{1}{2}$

Using the Clebsch-Gordan coefficient table.

$l \times s$ / $1 \times \frac{1}{2}$		$3/2$, $-\frac{1}{2}$	$1/2$, $-\frac{1}{2}$
0	$-\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{3}$

$$\Rightarrow \underset{l\ m_l}{|1\ 0\rangle} \underset{s\ m_s}{|\tfrac{1}{2}\ -\tfrac{1}{2}\rangle} = \sqrt{\frac{2}{3}} \left|\frac{3}{2}\ -\frac{1}{2}\right\rangle + \sqrt{\frac{1}{3}} \left|\frac{1}{2}\ -\frac{1}{2}\right\rangle$$

So $P_{j=\frac{3}{2}} = \frac{2}{3}$, $P_{j=\frac{1}{2}} = \frac{1}{3}$

$\Downarrow$

$\left(J^2 = \frac{3}{2} \cdot \frac{5}{2} \cdot \hbar^2 = \frac{15}{4}\hbar^2\right)$ $\quad \left(J^2 = \frac{1}{2} \cdot \frac{3}{2}\hbar^2 = \frac{3}{4}\hbar^2\right)$.

4. (a) $$\psi(x_1, x_2) = \frac{2}{a}\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)$$

$\uparrow\downarrow$

$$\times\, \chi_{\uparrow\downarrow}$$

singlet.

(b)

$\uparrow$ / $\uparrow$ or $\downarrow$ / $\uparrow$ ($\downarrow\uparrow$).

$$\psi(x_1, x_2) = \frac{\sqrt{2}}{a}\left(\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right) + \sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)\right)$$

$$\times\, \chi_{\uparrow\downarrow}$$

singlet

or $$\psi(x_1, x_2) = \frac{\sqrt{2}}{a}\left(\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right) - \sin\left(\frac{2\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)\right)$$

$$\times\, \chi_{\uparrow\uparrow}$$

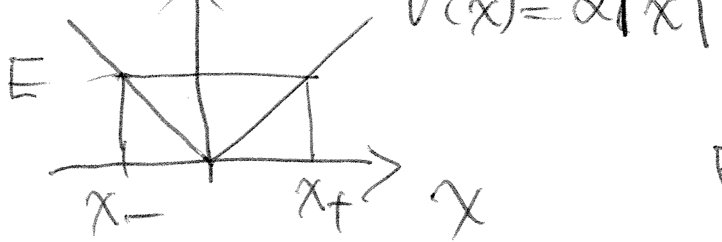
triplet.

So first excited state is degenerate.

(c) Since coulomb interaction is repulsive between electrons, energy of each level will increase.

For the 1st excited state, the spatially symmetric waveftn will have more overlap ~~in~~ between the two electrons, and so it will have more increase in energy than the spatially asymmetric state. In other words, the ~~triple~~ singlet state is now higher in energy, and the degeneracy is broken for the 1st excited state.

5.



$$E = \alpha|x_\pm|$$

$$\Rightarrow x_\pm = \pm\frac{E}{\alpha}.$$

$$\int_{x_-}^{x_+} p(x)\,dx = (n-\tfrac{1}{2})\pi\hbar \Rightarrow \int_{-\frac{E}{\alpha}}^{\frac{E}{\alpha}} \sqrt{2m(E-\alpha|x|)}\,dx = (n-\tfrac{1}{2})\pi\hbar$$

~~Final~~ ~~6~~ ~~n=1, for ground state~~

$$\Rightarrow \sqrt{2mE} \int_{-\frac{E}{\alpha}}^{\frac{E}{\alpha}} \sqrt{1-\frac{\alpha|x|}{E}}\, dx = (n-\tfrac{1}{2})\pi\hbar$$

$$\Rightarrow \sqrt{2mE} \int_0^{\cancel{\frac{E}{\alpha}}\,1} \sqrt{1-t}\ \frac{E}{\alpha} dt = (n-\tfrac{1}{2})\pi\hbar$$

$$\uparrow \quad \frac{\alpha x}{E} = t \Rightarrow dx = \frac{E}{\alpha} dt$$

$$\Rightarrow -\frac{2}{3}(1-t)^{\frac{3}{2}}\Big|_0^1 \times \frac{\sqrt{2mE}\ E}{\alpha} = (n-\tfrac{1}{2})\pi\hbar$$

$$\Rightarrow E^{\frac{3}{2}} \cdot \frac{2}{3}\frac{\sqrt{2m}}{\alpha} = (n-\tfrac{1}{2})\pi\hbar$$

$$\Rightarrow E^{\frac{3}{2}} = \frac{3}{2\sqrt{2}} \cdot \frac{1}{2}\pi\hbar\frac{\alpha}{\sqrt{m}}$$

n = 1 for ground state

$$\Rightarrow E = \left(\frac{3}{4\sqrt{2}}\frac{\pi\alpha\hbar}{\sqrt{m}}\right)^{\frac{2}{3}}$$

6. (a) $\psi(x,y) = \frac{2}{a}\sin(\frac{\pi}{a}x)\sin(\frac{2\pi}{a}y)$

and ~~or~~ $\frac{2}{a}\sin(\frac{2\pi}{a}x)\sin(\frac{\pi}{a}y)$

with $E = \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}(1^2+2^2)$

$= \frac{\hbar^2}{2m}\frac{\pi^2}{a^2}\cdot 5$

(b) with $|a\rangle = \frac{2}{a}\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}y\right)$

$|b\rangle = \frac{2}{a}\sin\left(\frac{2\pi}{a}x\right)\sin\left(\frac{\pi}{a}y\right)$

$$H'_{aa} = \langle a|H'|a\rangle$$

$$= \left(\frac{2}{a}\right)^2 \int_x\int_y \sin^2\left(\frac{\pi}{a}x\right)\sin^2\left(\frac{2\pi}{a}y\right) a^2 V_0\,\delta\left(x-\frac{a}{4}\right) \times \delta\left(y-\frac{3a}{4}\right) \times dx\,dy$$

$$= 4V_0 \sin^2\left(\frac{\pi}{4}\right)\sin^2\left(\frac{3\pi}{2}\right)$$

$$= 4V_0 \cdot \frac{1}{2} = \underline{2V_0}$$

$$H'_{ab} = \langle a|H'|b\rangle$$

$$= 4V_0 \int_x\int_y \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}y\right) \sin\left(\frac{2\pi}{a}x\right)\sin\left(\frac{\pi}{a}y\right)\delta\left(x-\frac{a}{4}\right)\delta\left(x-\frac{3}{4}a\right) \times dx\,dy$$

$$= 4V_0 \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right) \sin\left(\frac{3}{2}\pi\right)\sin\left(\frac{3}{4}\pi\right)$$

$$= 4V_0 \cdot \frac{1}{\sqrt{2}} \cdot (-1) \cdot \frac{1}{\sqrt{2}} = \underline{-2V_0}$$

$$H'_{ba} = (H'_{ab})^* = -2V_0$$

$$H'_{bb} = 4V_0 \sin^2\left(\frac{\pi}{2}\right) \sin^2\left(\frac{3}{4}\pi\right)$$

$$= 4V_0 \cdot 1 \cdot \frac{1}{2} = 2V_0$$

So $H' \equiv 2V_0 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$.

with $H' \begin{pmatrix} a \\ b \end{pmatrix} = 2V_0 \cdot \lambda \begin{pmatrix} a \\ b \end{pmatrix}$

$$\Rightarrow \quad \underbrace{\begin{pmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{pmatrix}}\begin{pmatrix} a \\ b \end{pmatrix} = 0$$

$$\Rightarrow \det[\quad] = 0 \Rightarrow (\lambda-1)^2 - 1 = 0$$

$$\Rightarrow \lambda - 1 = \pm 1$$

$$\Rightarrow \lambda = 2, \text{ or } 0.$$

So first order energy corrections are

~~$2V_0$~~ $4V_0$, and 0.

7. We need to consider the matrix element ($H = -q\vec{E}\cdot\vec{r} = -qEx$ $\Rightarrow H_{ij} = -qE\, x_{ij}$).

$\langle \psi_{100} | x | \psi_{200} \rangle$

$\langle \psi_{100} | x | \psi_{210} \rangle$, $\langle \psi_{100} | x | \psi_{211} \rangle$

and $\langle \psi_{100} | x | \psi_{21-1} \rangle$.

Among these, if we consider the angular part, with $x = r\sin\theta\cos\phi$

$\langle Y_0^0 | \sin\theta\cos\phi | Y_0^0 \rangle = 0$.

$\left[\frac{1}{4\pi} \iint \sin\theta\cos\phi \sin\theta \, d\theta\, d\phi \right]$

$\langle Y_0^0 | \sin\theta\cos\phi | Y_1^0 \rangle = \text{const} \int_\theta \sin\theta\cos\phi \cdot \cos\theta \cdot \sin\theta\, d\theta\, d\varphi$

$= \int_\theta [\;]\, d\theta \; \cancel{\int_{\varphi=0}^{2\pi} \cos\phi\, d\phi}^{\,0}$

$= 0$

$\langle Y_0^0 | \sin\theta\cos\phi | Y_1^{\pm1} \rangle = \text{const} \int \sin\theta\cos\phi\sin\theta e^{\pm i\phi} \cdot \sin\theta\, d\theta\, d\varphi$

$= \text{const} \int_\theta \sin^3\theta\, d\theta \int_\varphi \cos\phi\, e^{\pm i\phi}\, d\phi$

$\neq 0$

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So the only non-vanishing matrix elements are

$\langle \psi_{100} | x | \psi_{211} \rangle$ and $\langle \psi_{100} | x | \psi_{21-1} \rangle$.

3wt

~~And~~ $\langle \psi_{100} | x | \psi_{200} \rangle$ and $\langle \psi_{100} | x | \psi_{210} \rangle$

are zero.

That is $100 \leftrightarrow 200$

$100 \leftrightarrow 210$ transitions

are forbidden.