Quantum Mechanics and Atomic Physics

Lecture 14:

Angular Momentum Operators

http://www.physics.rutgers.edu/ugrad/361

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Last time

- Solved for S.E. in 3D using "particle in a box" example
- Using *separation of variables* we found that the total wavefunction is:

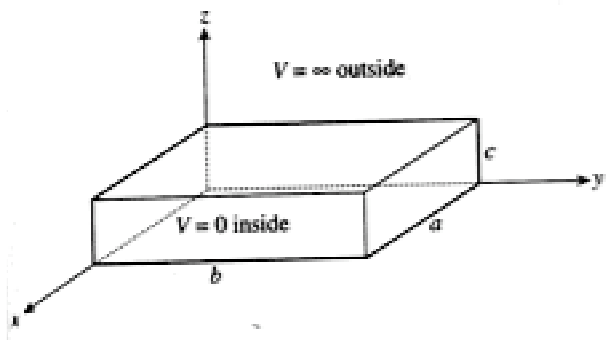

FIGURE 6.1 Infinite potential box.

$$\psi(x,y,z) = XYZ = A \sin\frac{n_x \pi x}{a} \sin\frac{n_y \pi}{b} y \sin\frac{n_z \pi}{c} z$$

$$n_x = 1,2,3,\ldots \quad , n_y = 1,2,3,\ldots \quad , n_z = 1,2,3,\ldots$$

$$A = \sqrt{\frac{8}{abc}} = \sqrt{\frac{8}{V}}$$

- Quantization of energy is:

$$E = \frac{\pi^2\hbar^2}{2m}\left[\frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2}\right]$$

- And found that energy degeneracy increases with energy

Spherical Coordinates

- One of our goals is to solve S.E. in 3D for the Coulomb potential (1/r) which describes the hydrogen atom
- To do this we need to use spherical coordinates

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

or

$$r = \sqrt{x^2+y^2+z^2}$$
$$\theta = \cos^{-1}(z/r)$$
$$\varphi = \tan^{-1}(y/x)$$

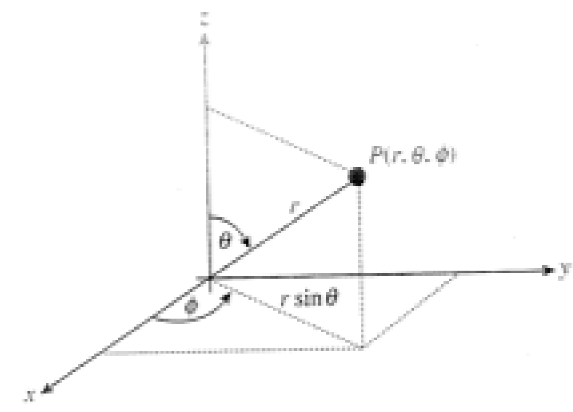

FIGURE 6.4 Spherical coordinates.

Reed: Chapter 6

Spherical Coordinates II

$$\hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z}$$

$$\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}$$

$$\Rightarrow \frac{\partial\hat{r}}{\partial\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} = \hat{\theta}$$

Similarly, $\dfrac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$, $\dfrac{\partial\hat{\phi}}{\partial\theta} = 0$

$$\frac{\partial\hat{r}}{\partial\phi} = \sin\theta\,\hat{\phi}, \quad \frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\,\hat{\phi}, \quad \frac{\partial\hat{\phi}}{\partial\phi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$$

But $\dfrac{\partial\hat{r}}{\partial r} = \dfrac{\partial\hat{\theta}}{\partial r} = \dfrac{\partial\hat{\phi}}{\partial r} = 0$

Gradient and Laplacian Operator in spherical coordinates

$$\nabla \equiv \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \varphi^2}$$

- We need this for defining angular momentum….

Angular Momentum

- Recall that angular momentum is:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{p} \Rightarrow \vec{p}_{op} = -i\hbar \vec{\nabla} \quad , \quad \vec{r} \Rightarrow \vec{r}_{op} = (x, y, z)$$

$$\vec{L}_{op} = \vec{r}_{op} \times \vec{p}_{op} = -i\hbar(\vec{r} \times \vec{\nabla})$$

- It can be easily shown that in Cartesian coordinates:

$$L_x \equiv -i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$L_y \equiv -i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$L_z \equiv -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Note:
No "x" appears in L_x,
No "y" appears in L_y,
And no "z" appears in L_z.

Commutators of x,y,z components of L

- For example:

$$[L_x, L_y] = L_x L_y - L_y L_x \qquad \text{(commutator)}$$

$$L_x L_y = \left\{-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right\}\left\{-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\}$$

$$= -\hbar^2\left(yz\frac{\partial}{\partial z}\frac{\partial}{\partial x} + y\frac{\partial}{\partial x} - yx\frac{\partial^2}{\partial z^2} - z^2\frac{\partial}{\partial y}\frac{\partial}{\partial x} + zx\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right)$$

$$L_y L_x = \left\{-i\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right\}\left\{-i\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right\}$$

$$= -\hbar^2\left(zy\frac{\partial}{\partial x}\frac{\partial}{\partial z} - z^2\frac{\partial}{\partial x}\frac{\partial}{\partial y} - xy\frac{\partial^2}{\partial z^2} + xz\frac{\partial}{\partial z}\frac{\partial}{\partial y} + x\frac{\partial}{\partial y}\right)$$

$$\Rightarrow L_x L_y - L_y L_x = -\hbar^2\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) \qquad \text{(all other terms cancel)}$$

So, $$[L_x, L_y] = -\hbar^2\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

But

$$i\hbar L_z = i\hbar\left(-i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\right)$$

$$= +\hbar^2\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

$$= -\hbar^2\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)$$

So,

$$[L_x, L_y] = i\hbar L_z$$

and similarly:

$$[L_y, L_z] = i\hbar L_x$$

$$[L_z, L_x] = i\hbar L_y.$$

What does this mean?

- The components of L do not commute with each other!
- No simultaneous eigenstates!
- If you measure $L_x \Rightarrow$ get a certain value
- Next, measure $L_y \Rightarrow$ get a certain value
- Measure L_x again $\Rightarrow$ in general, you won't get the same value as before!
- We will revisit this in Chapter 8

L operator in spherical coordinates

$$\vec{L}_{op} = -i\hbar(\vec{r}\times\vec{\nabla})$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\vec{L}_{op} = -i\hbar\left[r\hat{r} \times \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\right]$$

$$= -i\hbar\left[\cancelto{0}{\hat{r}\times\hat{r}}\left(r\frac{\partial}{\partial r}\right) + \underbrace{\hat{r}\times\hat{\theta}}_{=\hat{\varphi}}\, r\frac{1}{r}\frac{\partial}{\partial\theta} + \underbrace{\hat{r}\times\hat{\varphi}}_{=-\hat{\theta}}\, r\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

$$\vec{L}_{op} = -i\hbar\left[\hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right]$$

Now the x,y,z components in spherical coordinates

- First the z component:

$$(L_z)_{op} = \hat{z} \cdot \vec{L}_{op} \quad , \quad (L_x)_{op} = \hat{x} \cdot \vec{L}_{op} \quad , \quad (L_y)_{op} = \hat{y} \cdot \vec{L}_{op}$$

use:

$$\hat{x} = \sin\theta\cos\varphi\,\hat{r} + \cos\theta\cos\varphi\,\hat{\theta} - \sin\varphi\,\hat{\varphi}$$
$$\hat{y} = \sin\theta\sin\varphi\,\hat{r} + \cos\theta\sin\varphi\,\hat{\theta} + \cos\varphi\,\hat{\varphi}$$
$$\hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$$

$$(L_z)_{op} = (\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \cdot \left\{-i\hbar\left(\hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\right\}$$

$$= -i\hbar\,\sin\theta\,\hat{\theta}\cdot\hat{\theta}\,\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} = i\hbar\frac{\partial}{\partial\varphi}$$

$$(L_z)_{op} = -i\hbar\frac{\partial}{\partial\varphi}$$

use $\hat{\theta}\cdot\hat{\theta} = 1$, all others = zero

- Similarly,

$$(L_x)_{\theta\varphi} = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \cot\theta\cos\varphi\frac{\partial}{\partial\varphi}\right)$$

$$(L_y)_{\theta\varphi} = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \cot\theta\sin\varphi\frac{\partial}{\partial\varphi}\right)$$

  - I leave it for you as an exercise to derive these

- Later, we will need to use the expression for L_z

What about important quantity L_{op}^2?

$$\vec{L}_{op}\cdot\vec{L}_{op} = L_{op}^2 = \left[-i\hbar\left\{\hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right\}\right]\cdot\left[-i\hbar\left\{\hat{\varphi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right\}\right]$$

use $\hat{\varphi}\cdot\hat{\theta} = 0$

and $\frac{\partial\hat{\varphi}}{\partial\varphi} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$, $\frac{\partial\hat{\varphi}}{\partial\theta} = 0$, $\frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$, $\frac{\partial\hat{\theta}}{\partial\varphi} = \cos\theta\,\hat{\varphi}$

$$\Rightarrow L_{op}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

compare to:

$$\vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

$$\Rightarrow \vec{\nabla}^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) - \frac{1}{\hbar^2 r^2}\cdot L_{op}^2$$

Let's relate this to the Hamiltonian

$$H_{op} = \frac{-\hbar^2}{2m} \nabla^2 + V(r,\theta,\varphi)$$

$$\Rightarrow H_{op} = \frac{-\hbar^2}{2mr^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{L_{op}^2}{2mr^2} + V(r,\theta,\varphi)$$

Central Potentials: V(r)

- The S.E. for a particle of mass μ moving in a central potential V(r):

$$-\frac{\hbar^2}{2\mu}\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2}\frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}\right)\right\}\psi$$

$$+V\psi=E\psi$$

  - I've replaced m with μ (to not be confused with new variable I will introduce shortly m_ℓ, and in reality it should be expressed as the reduced mass μ)

- Again, let's use separation of variables, like we did last time:

$$\psi(r,\theta,\varphi)=R(r)\,Y(\theta,\varphi)$$

$$-\frac{\hbar^2}{2\mu}\left\{\frac{Y}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{R}{r^2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\varphi^2}\right)\right]\right\}$$

$$+V(R\cdot Y)=E\cdot R\cdot Y.$$

Multiply by $\frac{r^2}{RY}$

$$-\frac{\hbar^2}{2\mu}\left\{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right\}+\left[V-E\right]r^2$$

$$=\frac{\hbar^2}{2\mu}\left[\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{Y\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\varphi^2}\right)\right]$$

- Some things to note about this equation …

Separation of variables

$$\left.\begin{array}{l} LHS \Rightarrow f(r) \\ RHS \Rightarrow g(\theta,\varphi) \end{array}\right\} \quad f(r) = g(\theta,\varphi) = \text{const.}$$

- (we will come back to this equation)

- Let's call the constant:

$$\text{const.} = -\frac{\alpha\hbar^2}{2\mu} \qquad \alpha \text{ is also a const.}$$

- Let's rewrite the RHS:

$$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\varphi^2}\right)\right]=\alpha Y$$

So, full wavefunction is:
$\Psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$

$$Y(\theta,\varphi)=\Theta(\theta)\,\Phi(\varphi)$$

$$\Rightarrow\ -\frac{\Phi}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{\Theta}{\sin^2\theta}\frac{d^2\Phi}{d\varphi^2}=\alpha\,\Theta\Phi$$

Multiply by $\dfrac{\sin^2\theta}{\Theta\Phi}$

$$\Rightarrow\ \frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\alpha\sin^2\theta=\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}$$

$$\Rightarrow \underbrace{\underbrace{\frac{\sin\theta}{\Theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) - \alpha\sin^2\theta}_{\text{function of } \theta} = \underbrace{\frac{1}{\Phi}\frac{d^2\Phi}{d\varphi^2}}_{\text{function of } \varphi}}_{\text{equal to const.}}$$

- Let's call the constant $(-m_\ell^2)$
- We will discover later that this is called the **mangetic quantum number**

Solutions to $\Phi(\phi)$

$$\Rightarrow \quad \frac{d^2\Phi}{d\varphi^2} = -m_\ell^2\,\Phi$$

$$\Phi(\varphi) = A\,e^{im_\ell\varphi}$$

You will see later that we don't need the $e^{-im\phi}$ term

Normalize:

$$1 = \int_0^{2\pi} \Phi^*\Phi\,d\varphi = \int_0^{2\pi} A^2 e^{-im_\ell\varphi} e^{+im_\ell\varphi}\,d\varphi$$

$$= A^2 \cdot 2\pi \quad \Rightarrow \quad A = \sqrt{\frac{1}{2\pi}}$$

$$\Rightarrow \quad \Phi(\varphi) = \sqrt{\frac{1}{2\pi}}\, e^{i m_\ell \varphi}$$ φ ranges from $0 \to 2\pi$

- Single-valuedness requires that: $\Phi(\varphi) = \Phi(\varphi + 2\pi)$

Φ is cyclic

$$\sqrt{\frac{1}{2\pi}}\, e^{i m_\ell \varphi} = \sqrt{\frac{1}{2\pi}}\, e^{i m_\ell \varphi}\, e^{i m_\ell 2\pi}$$

$$\Rightarrow \quad e^{i m_\ell 2\pi} = 1$$

$$\Rightarrow \quad \cos 2\pi m_\ell + i \sin 2\pi m_\ell = 1$$

$$\Rightarrow \quad \cos 2\pi m_\ell = 1 \quad \underline{\text{and}} \quad \sin 2\pi m_\ell = 0$$

So, $2\pi m_\ell$ must be multiple of 2π

$$m_\ell = 0, \pm 1, \pm 2, \ldots$$

- We find that the mangnetic quantum number is quantized!

Let's go back to "separated" S.E.

$$\alpha = \frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) - \frac{2\mu}{\hbar^2} r^2 [V-E]$$

$$= -\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) \underbrace{- \frac{1}{\Phi \sin^2\theta}\frac{d^2\Phi}{d\varphi^2}}_{+\frac{m_\ell^2}{\sin^2\theta}}$$

$$\Rightarrow \frac{1}{R}\frac{d}{dr}\left(r^2 \frac{\partial R}{\partial r}\right) - \frac{2\mu}{\hbar^2} r^2 [V-E] = \frac{m_\ell^2}{\sin^2\theta} - \frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right)$$

- RHS depends on θ
- LHS depends on r
- Equal to a constant
  - Let's call it $\ell(\ell+1)$ …. Which is also equal to what we've been calling α!

$$\Rightarrow \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1) - \frac{m_\ell^2}{\sin^2\theta}\right]\Theta = 0$$

$$\Rightarrow \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \left[\frac{2\mu}{\hbar^2}(V-E) - \frac{\ell(\ell+1)}{r^2}\right]R = 0$$

- Solutions to $\Theta(\theta)$:
  - They are associated Legendre Polynomials (see Chapter 6 for full expression):

$$P_{\ell m_\ell}(\cos\theta)$$

  - A few of them are:

$$P_{00} = 1, \quad P_{10} = \cos\theta, \quad P_{11} = \sin\theta, \quad P_{1-1} = -\sin\theta$$

$$P_{20} = \frac{1}{2}(3\cos^2\theta - 1)$$

  - Note: special cases of m_ℓ=0 are called Regular Legendre Polynomials

Spherical Harmonics

- We usually deal with a combined angular dependence $Y(\theta,\phi)=\Theta(\theta)\Phi(\phi)$
- These are called Spherical Harmonics (see Chapter 6 for full expression): $Y_{\ell m_\ell}(\theta,\varphi)$
- And is ℓ called the orbital angular momentum quantum number

Table 6.1 Spherical Harmonics

$Y_{0,0}(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$	$Y_{1,0}(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta$
$Y_{1,\pm1}(\theta,\phi) = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\phi}$	$Y_{2,0}(\theta,\phi) = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$
$Y_{2,\pm1}(\theta,\phi) = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\phi}$	$Y_{2,\pm2}(\theta,\phi) = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\phi}$
$Y_{3,0}(\theta,\phi) = \sqrt{\frac{7}{16\pi}}(5\cos^3\theta - 3\cos\theta)$	$Y_{3,\pm1}(\theta,\phi) = \mp\sqrt{\frac{21}{64\pi}}\sin\theta(5\cos^2\theta - 1)\, e^{\pm i\phi}$
$Y_{3,\pm2}(\theta,\phi) = \sqrt{\frac{105}{32\pi}}\sin^2\theta\cos\theta\, e^{\pm 2i\phi}$	$Y_{3,\pm3}(\theta,\phi) = \mp\sqrt{\frac{35}{64\pi}}\sin^3\theta\, e^{\pm 3i\phi}$

Radial Solutions

- Solutions to R(r), depend on the potential energy
  - We will revisit this next week

Let's go back to:

Recall:

$$\boxed{-\frac{\hbar^2}{2\mu}\left\{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right\} + [V-E]r^2 = \frac{\hbar^2}{2\mu}\left[\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\left(\frac{\partial^2 Y}{\partial\varphi^2}\right)\right]}$$

$$\boxed{L_{op}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]}$$

$$f(r) = g(\theta,\varphi) = \text{const}$$

$$\Rightarrow -\frac{\hbar^2}{2\mu}\left\{\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)\right\} + [V-E]r^2$$

$$= \text{const} = -\frac{1}{2\mu Y}\left[L_{op}^2 Y\right] \qquad \text{const} = -\frac{\alpha\hbar^2}{2\mu}$$

$$\Rightarrow \quad -\frac{1}{2\mu Y}\left[L_{op}^2 Y\right] = -\frac{\alpha\hbar^2}{2\mu}$$

$$\Rightarrow \quad L_{op}^2 Y = \hbar^2 \alpha Y = \left[\hbar^2 (\ell(\ell+1))\right] Y$$

$$\Rightarrow \quad L = \sqrt{\ell(\ell+1)}\,\hbar \qquad \ell = 0, 1, 2, \ldots$$

So, $\alpha = \ell(\ell+1)$ and $\ell = 0, 1, 2, \ldots$

Total angular momentum is quantized!

What about L_z?

$$L_z = -i\hbar \frac{\partial}{\partial \varphi}$$

$$\Rightarrow \quad L_z \Phi(\varphi) = -i\hbar \frac{\partial}{\partial \varphi}\left(A e^{i m_\ell \varphi}\right)$$

$$= -i^2 m_\ell \hbar \left(A e^{i m_\ell \varphi}\right)$$

$$= m_\ell \hbar \Phi$$

$$\Rightarrow \quad L_z = m_\ell \hbar \qquad \text{where } |m_\ell| \le \ell$$

$$\text{since} \quad L_z \le |\vec{L}| \quad \Rightarrow \quad m_\ell \le \sqrt{\ell(\ell+1)}$$

$$\Rightarrow \quad |m_\ell| \le \ell$$

- So, solutions exist only if ℓ is an integer <u>and</u> $\ell \geq m_\ell$

Summary/Announcements

- For central potentials (V depends only on r and not on any angles), the angular solution is always, the spherical harmonics ($Y_{l,m}(\theta,\phi)$), well defined quantum angular momentum quantum numbers, *l* and *m*.
- Next time: More about angular momentum and radial solutions for central potentials
- Next homework due on Monday Nov 7