Quantum Mechanics and Atomic Physics 750:361

Prof. Sean Oh, Fall 2011

HW #8

Due date: Monday, Nov. 7, 2011, at the beginning of class

1. Reed, Prob. 6-6

Using the method demonstrated in equation (6.3.9), evaluate the other three terms in equation (6.3.8), and hence verify equations (6.3.10) and (6.3.11) for $L_{op}^2$.

The first term in (6.3.8) is, using equation (6.2.7)

$$\left(\boldsymbol{\phi}\frac{\partial}{\partial\theta}\right)\bullet\left(\boldsymbol{\phi}\frac{\partial}{\partial\theta}\right) = \boldsymbol{\phi}\bullet\left(\frac{\partial\boldsymbol{\phi}}{\partial\theta} + \boldsymbol{\phi}\frac{\partial^2}{\partial^2\theta}\right) = \frac{\partial^2}{\partial^2\theta}.$$

The second term yields zero:

$$\left(\boldsymbol{\phi}\frac{\partial}{\partial\theta}\right)\bullet\left(\boldsymbol{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) = \boldsymbol{\phi}\bullet\left(\frac{\partial\boldsymbol{\theta}}{\partial\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\boldsymbol{\theta}\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} + \boldsymbol{\theta}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\right)$$

$$= \boldsymbol{\phi}\bullet\left(-\boldsymbol{r}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} - \frac{\boldsymbol{\theta}\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} + \boldsymbol{\theta}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\theta\partial\phi}\right) = 0.$$

The fourth term gives

$$\left(\boldsymbol{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right)\bullet\left(\boldsymbol{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) = \boldsymbol{\theta}\frac{1}{\sin\theta}\bullet\left(\frac{\partial\boldsymbol{\theta}}{\partial\phi}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \boldsymbol{\theta}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right)$$

$$= \boldsymbol{\theta}\frac{1}{\sin\theta}\bullet\left(\boldsymbol{\phi}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\phi} + \boldsymbol{\theta}\frac{1}{\sin\theta}\frac{\partial^2}{\partial\phi^2}\right) = \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Hence $L_{op}^2$ emerges as

$$L_{op}^2 = -\hbar^2\left[\frac{\partial^2}{\partial^2\theta} + \cot\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right].$$

2. Reed, Prob. 6-7

Following the method that was used to derive the expression for $(L_z)_{op}$ in section 6.3, show that the operators for the x and y components of angular momentum in spherical coordinates are given by

$$L_x \equiv -\iota\hbar\left(-\sin\phi\,\frac{\partial}{\partial\theta} - \cot\theta\cos\phi\,\frac{\partial}{\partial\phi}\right)$$

and

$$L_y \equiv -\iota\hbar\left(\cos\phi\,\frac{\partial}{\partial\theta} - \cot\theta\,\sin\phi\,\frac{\partial}{\partial\phi}\right).$$

Here we have

$$L_x \equiv \boldsymbol{x}\bullet\mathbf{L}_{op} = -\iota\hbar\left(\sin\theta\cos\phi\,\boldsymbol{r} + \cos\theta\cos\phi\,\boldsymbol{\theta} - \sin\phi\,\boldsymbol{\phi}\right)\bullet\left\{\boldsymbol{\phi}\frac{\partial}{\partial\theta} - \boldsymbol{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right\}$$

$$= -\iota\hbar\left\{-\cot\theta\cos\phi\frac{\partial}{\partial\phi} - \sin\phi\frac{\partial}{\partial\theta}\right\}.$$

and

$$L_y \equiv \boldsymbol{y}\bullet\mathbf{L}_{op} = -\iota\hbar(\sin\theta\sin\phi\,\boldsymbol{r} + \cos\theta\sin\phi\,\boldsymbol{\theta} + \cos\phi\,\boldsymbol{\phi})\bullet\left\{\boldsymbol{\phi}\frac{\partial}{\partial\theta} - \boldsymbol{\theta}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right\}$$

$$= -\iota\hbar\left\{-\cot\theta\sin\phi\frac{\partial}{\partial\phi} + \cos\phi\frac{\partial}{\partial\theta}\right\}.$$

3. Reed, Prob. 6-11

Prove that $[L_x, p_x] \equiv 0$, $[L_x, p_y] \equiv (\iota\hbar)p_z$, and $[L_x, p_z] \equiv -(\iota\hbar)p_y$. Permuting the coordinates leads also to the identities $[L_y, p_x] \equiv -(\iota\hbar)p_z$, $[L_y, p_y] \equiv 0$, $[L_y, p_z] \equiv (\iota\hbar)p_x$, $[L_z, p_x] \equiv (\iota\hbar)p_y$, $[L_z, p_y] \equiv -(\iota\hbar)p_x$, and $[L_z, p_z] \equiv 0$.

The momentum operators are

$$(p_x, p_y, p_z) \equiv -\iota\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

The angular momentum operators are

$$L_x \equiv -\iota\hbar\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right),$$

$$L_y \equiv -\iota\hbar\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right),$$

and

$$L_z \equiv -\iota\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right).$$

Apply $[L_x, p_x]$ to some dummy wavefunction, and use the subscript notation as in the solution of problem 6-10 above to denote derivatives:

$$[L_x, p_x]\psi = (L_x p_x - p_x L_x)\psi$$

$$= (\iota\hbar)^2\left(y\frac{\partial\psi_x}{\partial z} - z\frac{\partial\psi_x}{\partial y}\right) - (\iota\hbar)^2\frac{\partial}{\partial x}\left(y\psi_z - z\psi_y\right)$$

$$= (\iota\hbar)^2\left(y\psi_{zx} - z\psi_{yx} - y\psi_{xz} + z\psi_{xy}\right) = 0.$$

For $[L_x, p_y]$:

$$[L_x, p_y]\psi = (L_x p_y - p_y L_x)\psi$$

$$= (\iota\hbar)^2\left(y\frac{\partial\psi_y}{\partial z} - z\frac{\partial\psi_y}{\partial y}\right) - (\iota\hbar)^2\frac{\partial}{\partial y}\left(y\psi_z - z\psi_y\right)$$

$$= (\iota\hbar)^2\left(y\psi_{zy} - z\psi_{yy} - \psi_z - y\psi_{yz} + z\psi_{yy}\right) = -(\iota\hbar)^2\frac{\partial\psi}{\partial z}$$

$$= (\iota\hbar)\left(-\iota\hbar\frac{\partial}{\partial z}\right)\psi = \iota\hbar p_z\psi.$$

For $[L_x, p_z]$:

$$[L_x, p_z]\psi = (L_x p_z - p_z L_x)\psi$$

$$= (\iota\hbar)^2\left(y\frac{\partial\psi_z}{\partial z} - z\frac{\partial\psi_z}{\partial y}\right) - (\iota\hbar)^2\frac{\partial}{\partial z}\left(y\psi_z - z\psi_y\right)$$

$$= (\iota\hbar)^2\left(y\psi_{zz} - z\psi_{yz} - y\psi_{zz} + \psi_y + z\psi_{zy}\right) = (\iota\hbar)^2\frac{\partial\psi}{\partial y}$$

$$= -(\iota\hbar)\left(-\iota\hbar\frac{\partial}{\partial y}\right)\psi = -\iota\hbar p_y\psi.$$

The other results cited in the question can be similarly deduced, or established more easily by cyclically permuting x, y, and z.

4. Reed, Prob. 7-2

Derive an expression for $\langle r^2 \rangle$ for a particle in the $\ell = 0$ state of the infinite spherical well.

For the $\ell = 0$ state of the infinite spherical well, the wavefunctions are

$$\psi_{n00} = \frac{1}{\sqrt{2\pi a}\, r} \sin\left(\frac{n\pi r}{a}\right).$$

$\langle r^2 \rangle$ is given by

$$\langle r^2 \rangle = \int_{r=0}^{a}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \psi_{n00}(r^2)\psi_{n00}\, r^2 \sin\theta\, dr\, d\theta\, d\phi = \frac{(4\pi)}{2\pi a}\int_0^a r^2 \sin^2\left(\frac{n\pi r}{a}\right) dr.$$

where the factor of $4\pi$ comes from the integrals over $\theta$ and $\phi$.

The integral for $\langle r^2 \rangle$ is of exactly the same form as that for $\langle x^2 \rangle$ in the infinite rectangular well (example 4.1) with the radius of the spherical well (a) playing the role of the width L of the rectangular well. Hence

$$\langle r^2 \rangle = \frac{a^2}{3}\left[1 - \frac{3}{2\pi^2 n^2}\right].$$

5. Reed, Prob. 7-4

Apply the results of problem 6-20 to the infinite spherical well wavefunctions ($\ell = 0$) of section 7.2. It is helpful to recall that for a mass m, $\langle p^2 \rangle = 2m\langle KE \rangle$ where $\langle KE \rangle$ is the kinetic energy. Is the uncertainty principle satisfied?

The wavefunction is

$$\psi_{n00} = \frac{1}{\sqrt{2\pi a}\, r} \sin\left(\frac{n\pi r}{a}\right).$$

From problem 6-20, $\Delta r \Delta p = \sqrt{\langle r^2 \rangle \langle p^2 \rangle}$. The mean value of (radius squared) is given by

$$\langle r^2 \rangle = \int_{r=0}^{a} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \psi(r^2\psi)\, r^2 \sin\theta\, d\phi\, d\theta\, dr.$$

The integrals over the angular coordinates yield $4\pi$:

$$\langle r^2 \rangle = \frac{2}{a} \int_0^a r^2 \sin^2\left(\frac{n\pi r}{a}\right) dr = a^2\left[\frac{1}{3} - \frac{1}{2n^2\pi^2}\right].$$

Now, $\langle p^2 \rangle = 2m\langle KE \rangle$. Since the potential energy for this system is zero, this gives

$$\langle p^2 \rangle = 2m\langle KE \rangle = 2m\langle E_n \rangle = \frac{\hbar^2 n^2 \pi^2}{a^2}.$$

These results give

$$\Delta r \Delta p = \sqrt{\langle r^2 \rangle \langle p^2 \rangle} = \sqrt{\left(a^2\left[\frac{1}{3} - \frac{1}{2n^2\pi^2}\right]\right)\left(\frac{\hbar^2 n^2 \pi^2}{a^2}\right)} = n\pi\hbar\sqrt{\frac{1}{3} - \frac{1}{2n^2\pi^2}}.$$

For $n = 1$, $\Delta r \Delta p = 1.670\hbar$; the uncertainty principle is satisfied.