Quantum Mechanics and Atomic Physics 750:361

Prof. Sean Oh, Fall 2011

HW #5

Due date: Monday, Oct. 10, 2011, at the beginning of class

1. Reed, Prob.3-28

In an experiment involving electron scattering from a finite rectangular well of depth 4 eV, it is found that electrons of energy 5 eV are completely "transmitted". What must be the width of the well? At what next higher energy can one expect to again observe T = 1?

From equation (3.10.15), resonant scattering occurs at

$$L = \frac{n\pi\hbar}{\sqrt{2m(E+V_o)}} = \frac{(1)\,\pi\,(1.055 \times 10^{-34})}{\sqrt{2(9.11 \times 10^{-31})(1.442 \times 10^{-18})}} = 2.045 \times 10^{-10} \text{ meters},$$

or L = 2.045 Å.

Resonant scattering occurs next for n = 2, hence

$$E = \frac{1}{2m}\left(\frac{n\pi\hbar}{L}\right)^2 - V_o = 5.126 \times 10^{-18} \text{ Joule} = 32 \text{ eV}.$$

2. Reed, Prob. 4-3

A particle of mass m has the hypothetical wavefunction $\psi(x) = Ae^{-\alpha x}$, $(0 \le x \le \infty;\ \psi = 0$, otherwise). Determine the normalization constant A in terms of α. Determine $\langle x \rangle$ and $\langle p \rangle$ in terms of α and m.

Normalization demands

$$\int_0^\infty \psi^2\,dx = A^2 \int_0^\infty e^{-2\alpha x}\,dx = 1.$$

This integral can be found in Appendix C, with the result

$$A^2 \int_0^\infty e^{-2\alpha x}\,dx = \frac{A^2}{2\alpha},$$

or $A = \sqrt{2\alpha}$. Hence we have

$$\langle x \rangle = A^2 \int_0^\infty x\,e^{-2\alpha x}\,dx = \frac{A^2}{4\alpha^2} = \frac{1}{2\alpha}$$

and

$$\langle p \rangle = A^2 \int_0^\infty e^{-\alpha x}(-\iota\hbar)\left(\frac{d}{dx}e^{-\alpha x}\right)dx = \iota\hbar\alpha A^2 \int_0^\infty e^{-2\alpha x}dx = \frac{\iota\hbar\alpha A^2}{2\alpha} = \iota\hbar\alpha.$$

3. Reed, Prob. 4-5

With $p = h\nu/c$, $\Delta p = h(\Delta\nu)/c$, hence

$$\Delta\nu = \frac{c}{h}\Delta p = \frac{3 \text{ x } 10^8}{6.626 \text{ x } 10^{-34}}(3.16 \text{ x } 10^7) = 80 \text{ sec}^{-1}.$$

A brief radio-wave pulse of photons is of duration $\tau = 0.001$ seconds. The pulse must then have a length of $\tau c = 3 \text{ x } 10^5$ meters, and, since an individual photon might be anywhere in the pulse, the uncertainty in the position of that photon will be $\Delta x = 3 \text{ x } 10^5$ meters. What is the corresponding uncertainty in the momentum of the photon? If the momentum and frequency of a photon are related by $p = h\nu/c$, what is the uncertainty in the frequency of the photon?

The uncertainty relation gives

$$\Delta p \ge \frac{\hbar}{2\Delta x} = \frac{1.055 \text{ x } 10^{-34}}{2(3 \text{ x } 10^5)} = 1.76 \text{ x } 10^{-40} \text{ kg(m/s)}$$

With $p = h\nu/c$, $\Delta p = h(\Delta\nu)/c$, hence

$$\Delta\nu = \frac{c}{h}\Delta p = \frac{3 \times 10^8}{6.626 \times 10^{-34}}(3.16 \times 10^7) = 80 \ \sec^{-1}.$$

4. Reed, Prob. 4-6

The ground state wavefunction of the harmonic oscillator potential (see Chapter 5) is given by

$$\psi_o(x) = \frac{\sqrt{\alpha}}{\pi^{1/4}} e^{-\alpha^2 x^2/2},$$

where α is a constant. Verify that $(\Delta x\, \Delta p) = \hbar/2$ for this state.

We have

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi_o^* x \psi_o \, dx = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} x e^{-\alpha^2 x^2} dx = 0, \text{ by symmetry}.$$

Similarly, with the help of Appendix C,

$$\langle x^2 \rangle = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2} dx = \frac{2\alpha}{\sqrt{\pi}} \int_{0}^{\infty} x^2 e^{-\alpha^2 x^2} dx = \frac{2\alpha}{\sqrt{\pi}} \frac{\sqrt{\pi}}{4\alpha^3} = \frac{1}{2\alpha^2}.$$

For $\langle p \rangle$ we find

$$\langle p \rangle = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2/2} \left(-\iota \hbar \frac{d}{dx} e^{-\alpha^2 x^2/2} \right) dx = -\frac{\iota \hbar \alpha^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} x e^{-\alpha^2 x^2} dx = 0, \text{ by symmetry}.$$

The computation of $\langle p^2 \rangle$ is more involved:

$$\langle p^2 \rangle = \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2/2} \left\{ -\hbar^2 \frac{d^2}{dx^2} (e^{-\alpha^2 x^2/2}) \right\} dx = \frac{\hbar^2 \alpha^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2/2} \left\{ \frac{d}{dx} (x e^{-\alpha^2 x^2/2}) \right\} dx$$

$$= \frac{\hbar^2 \alpha^3}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2/2} (e^{-\alpha^2 x^2/2} - x^2 \alpha^2 e^{-\alpha^2 x^2/2}) \, dx$$

$$= \frac{2\hbar^2 \alpha^3}{\sqrt{\pi}} \left\{ \int_{0}^{\infty} e^{-\alpha^2 x^2} dx - \alpha^2 \int_{0}^{\infty} x^2 e^{-\alpha^2 x^2} dx \right\} = \frac{2\hbar^2 \alpha^3}{\sqrt{\pi}} \left\{ \frac{\sqrt{\pi}}{2\alpha} - \frac{\alpha^2 \sqrt{\pi}}{4\alpha^3} \right\} = \frac{\hbar^2 \alpha^2}{2}.$$

These results give

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{2}\alpha}$$

and

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \frac{\hbar\alpha}{\sqrt{2}}.$$

Hence

$$\Delta x \Delta p = \frac{\hbar}{2}.$$

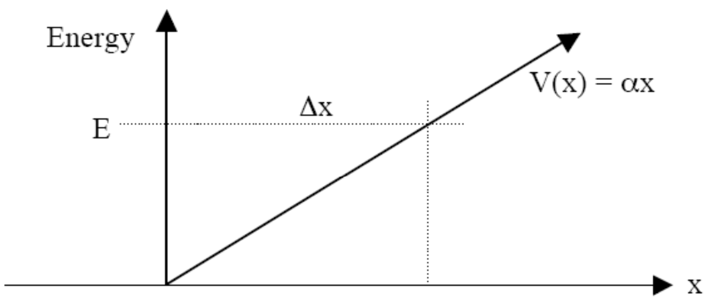
5. Reed, Prob. 4-10

Use the uncertainty principle to estimate the ground state energy of a particle of mass m moving in the *linear potential*

$$V(x) = \begin{cases} \infty & (x \le 0), \\ \alpha x & (x \ge 0). \end{cases}$$

HINT: Imagine some energy level E that cuts the potential at x = 0 and x = E/α. Take x ~ Δx and $p \sim \Delta p \sim \hbar / 2(\Delta x)$.

A sketch of the potential is shown below. An energy level E will cut the potential at x = E/α. Since we know that the wavefunction will decrease exponentially beyond this limit, we can assume that the bulk of it is confined to within 0 < x < (E/a), hence we can argue Δx ~ x.



Using the hints given, the energy of the system can be expressed as

$$E = \frac{p^2}{2m} + V(x) = \frac{\hbar^2}{8m(\Delta x)^2} + \alpha(\Delta x).$$

The minimum energy is found by computing the derivative of E with respect to Δx and equating it to zero:

$$\frac{dE}{d(\Delta x)} = -\frac{\hbar^2}{4m(\Delta x)^3} + \alpha = 0,$$

which gives

$$\Delta x = \left(\frac{\hbar^2}{4m\alpha}\right)^{1/3}.$$

Back-substituting this result into the expression for E gives

$$E = \frac{\hbar^2}{8m}\left(\frac{4m\alpha}{\hbar^2}\right)^{2/3} + \alpha\left(\frac{\hbar^2}{4m\alpha}\right)^{1/3} = \frac{3}{2^{4/3}}\alpha^{2/3}\left(\frac{\hbar^2}{2m}\right)^{1/3} = 1.191\,\alpha^{2/3}\left(\frac{\hbar^2}{2m}\right)^{1/3}.$$

6. Reed, Prob. 4-11

Consider a non-relativistic "free" particle (that is, one not subject to a potential) of mass m moving with speed v. The energy of this particle will be $mv^2/2 = p^2/2m$ where $p = mv$. Show that the uncertainty principle $\Delta x \Delta p \geq \hbar/2$ becomes $\Delta E \Delta t \geq \hbar/2$ in this case. According to Einstein, this mass has an energy equivalent given by $E = mc^2$; show also that the uncertainty principle can then be written in the form $\Delta m \Delta t \geq \hbar/2c^2$. A free neutron (one not bound within a nucleus) has a half-life against beta-decay of 10.25 minutes. Taking this time as an estimate of Δt, what is the corresponding uncertainty in the mass of the neutron?

Momentum is defined as $p = mv$. Writing v as $\Delta x/\Delta t$ gives this as

$$p = m\frac{\Delta x}{\Delta t} \quad \Rightarrow \quad \Delta x = \frac{p}{m}\Delta t.$$

Now take the derivative of $E = p^2/2m$ to give

$$\Delta E = \frac{2p\Delta p}{2m} = \frac{p}{m}\Delta p \quad \Rightarrow \quad \Delta p = \frac{m}{p}\Delta E.$$

Substituting both of these into the Uncertainty Principle gives

$$\Delta x \Delta p \geq \frac{\hbar}{2} \quad \Rightarrow \quad \left(\frac{p}{m}\Delta t\right)\left(\frac{m}{p}\Delta E\right) \geq \frac{\hbar}{2} \quad \Rightarrow \quad \Delta E \Delta t \geq \frac{\hbar}{2}.$$

From $E = mc^2$, $\Delta E = c^2 \Delta m$, hence

$$\Delta m \Delta t \geq \frac{\hbar}{2c^2}.$$

For the neutron, $\Delta t \sim 10.25$ minutes ~ 615 seconds. Hence

$$\Delta m \geq \frac{\hbar}{2c^2 \Delta t} \geq \frac{1.0546 \times 10^{-34} \text{ J-s}}{2(2.9979 \times 10^8 \text{ m/s})^2 (615 \text{ s})} \geq 9.540 \times 10^{-55} \text{ kg}.$$

Δm amounts to about 5.7 x 10^{-28} times the mass of the neutron itself: absolutely inconsequential.

7. Reed, Prob. 4-17.

Current versions of *string theory* hold that the most fundamental structures in the Universe may be "strings" of length L ~ 10^{-35} m vibrating in various energy states. The mass of these strings is taken to be the *Planck mass*, $m_p = \sqrt{\hbar c / G}$ where G is the Newtonian gravitational constant. Using the argument presented at the end of section 4.3, estimate the lowest possible energy for a vibrating string. Express your answer in giga-electron-volts (GeV = 10^9 eV). Do you think such energies are obtainable with current particle accelerators?

The argument in section 4.3 showed that for a particle of mass m trapped in an infinite rectangular well of width L, the lowest energy state must satisfy

$$E \geq \frac{\hbar^2}{8mL^2}.$$

Setting $m = \sqrt{\hbar c / G}$ gives

$$E \geq \frac{1}{8L^2}\sqrt{\frac{G}{c}}\hbar^{3/2}.$$

Putting in the numbers gives $E \geq 6.39 \times 10^8$ Joules, or $E \geq 4 \times 10^{18}$ GeV. The Fermilab accelerator can obtain energies of about 1 TeV (1 TeV = 10^{12} eV): string theory is far from being experimentally tested.