HW #2 Solution

**Problem 2-1**

Verify that if one has a number of independent solutions $y_1$ , $y_2$, … $y_n$ of the classical wave equation (all of pattern speed v), then a linear sum of them (equation 2.1.10) is also a solution.

Each independent solution $y_m$ (x) must satisfy

$$\frac{\partial^2 y_m}{\partial t^2} = v^2 \frac{\partial^2 y_m}{\partial x^2}.$$

A linear sum of $y_n$'s appears as

$$y = a_1 y_1 + a_2 y_2 + ... + a_n y_n = \sum a_n y_n .$$

Differentiating this linear sum and using the first equation above gives

$$\frac{\partial^2 y}{\partial x^2} = \sum a_n \frac{\partial^2 y_n}{\partial x^2} = v^2 \sum a_n \frac{\partial^2 y_n}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial t^2}.$$

**Problem 2-5**

Given the wavefunction $\psi(x) = Axe^{-kx}$ $(0 \le x \le \infty;\ k > 0)$, what value must A take in terms of k in order that $\psi$ be normalized? See Appendix C for the relevant integral.

We must have

$$\int_0^\infty \psi^2(x)dx = 1,$$

that is,

$$A^2 \int_0^\infty x^2 e^{-2kx} dx = 1.$$

This integral can be evaluated using the table of integrals in Appendix C; the result is

$$\frac{A^2}{4k^3} = 1,$$

or $A = 2k^{3/2}$.

**Problem 2-6**

Suppose that the wavefunction in problem 2-5 is known to be a solution of the Schrödinger equation for some energy E. What is the corresponding potential function V(x)?

We have

$$\psi(x) = Axe^{-kx}.$$

Hence

$$\frac{d\psi}{dx} = Ae^{-kx} - kAxe^{-kx}$$

and

$$\frac{d^2\psi}{dx^2} = (-2k + k^2x)Ae^{-kx}.$$

The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi.$$

Substituting the second derivative gives

$$-\frac{\hbar^2}{2m}(-2k + k^2x)Ae^{-kx} + V(x)Axe^{-kx} = EAxe^{-kx}.$$

Canceling the common factor of $Ae^{-kx}$ and dividing through by x leaves

$$V(x) = E + \frac{\hbar^2k^2}{2m} - \frac{\hbar^2k}{mx}.$$

**Problem 2-7**

The wavefunction $\psi(x) = Ae^{-kx^2}$ $(-\infty \le x \le \infty;\ k > 0)$ is known to be a solution of the Schrödinger equation for some energy E. What is the corresponding potential function V(x)? We will explore a potential like this in Chapter 5.

We need the second derivative of $\psi$:

$$\psi(x) = Ae^{-kx^2},$$

so

$$\frac{d\psi}{dx} = -(2kA)xe^{-kx^2},$$

hence

$$\frac{d^2\psi}{dx^2} = -(2kA)e^{-kx^2} + (4k^2A)x^2e^{-kx^2}.$$

The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi.$$

Substituting the second derivative gives

$$-\frac{\hbar^2}{2m}\left[-(2k) + (4k^2)x^2\right]Ae^{-kx^2} + V(x)Ae^{-kx^2} = EAe^{-kx^2}.$$

Canceling the common factor of $Ae^{-kx^2}$ leaves

$$V(x) = E + \frac{\hbar^2}{2m}\left(4k^2x^2 - 2k\right).$$

This is a more elaborate version of the *harmonic oscillator* potential of Chapter 5.

5. Is $\Psi(x,t) = Ae^{i(kx-wt)}$ square-integrable for $x \in (-\infty,\infty)$? Explain.

$$\int_{-\infty}^{\infty} \Psi^*\Psi dx = |A|^2 \int_{-\infty}^{\infty} e^{-i(kx-\omega t)}e^{i(kx-\omega t)}dx = |A|^2 \int_{-\infty}^{\infty} 1dx = \infty$$

So it is not square-integrable.