Problem 1-4

To establish the relationship between equation (1.2.13) and Figure 1.3 requires some knowledge of thermodynamics, which shows that the rate of energy escape (power) between wavelengths λ and $\lambda + d\lambda$ through a hole of area A in a blackbody cavity of temperature T is given by

$$P(\lambda)d\lambda \;=\; A\left\{\frac{2\pi hc^2}{\lambda^5\left(e^{hc/\lambda kt}-1\right)}\right\}d\lambda .$$

Consider a blackbody at T = 7000 K. What is the power emitted through a hole of area 1 m^2 between λ = 4000Å and 4001Å? Does your result accord with Figure 1.3?

Substituting the numbers (all MKS units) gives

$$\frac{hc}{\lambda kT} \;=\; \frac{(6.626 \times 10^{-34})(2.998 \times 10^{8})}{(4 \times 10^{-7})(1.381 \times 10^{-23})(7000)} \;=\; 5.137,$$

hence

$$P(\lambda)d\lambda \;=\; A\left\{\frac{2\pi hc^2}{\lambda^5\left(e^{hc/\lambda kt}-1\right)}\right\}d\lambda \;=\; (1)\left\{\frac{(2\pi)(6.626 \times 10^{-34})(2.998 \times 10^{8})^2}{(4 \times 10^{-7})^5\ (e^{5.137}-1)}\right\}(10^{-10}) \;=\; 21,590.$$

The units must be Watts; $P(\lambda)d\lambda$ = 21.59 kW, which accords with the Figure.

Problem 1-8

An electron transits between two energy levels separated by energy E, emitting a photon of wavelength λ in accordance with the Planck/Bohr relation $E = hc/\lambda$. (a) Show that if λ is given in units of Ångstroms, then $E = 12398/\lambda$ electron-volts. (b) As a consequence of some perturbing effect, the separation of the energy levels is altered by an amount dE electron-volts. Show that the corresponding change in the wavelength of the emitted photon is $d\lambda = -\lambda^2 dE/12398$ Ångstroms. If a transition leading to a photon of wavelength 4000Å is perturbed by dE = 0.01 eV, what is $d\lambda$?

(a)   We have

$$E = \frac{hc}{\lambda}.$$

If h, c, and λ are specified in MKS units, then E must emerge in Joules. If λ is given instead in Å (written as λ_A), it can be converted to meters by multiplying by 10^{10}, hence,

$$E = \frac{hc}{10^{10}\lambda_A} = \frac{(6.6261 \times 10^{34} J - \sec)(2.9979 \times 10^8 \, m/\sec)}{10^{10}\lambda_A \ (m)} = \frac{1.9864 \times 10^{-15}}{\lambda_A} \ \text{Joules}.$$

Now, 1 Joule $= 1.6022 \times 10^{-19}$ eV, hence

$$E_{eV} = \frac{1.9864 \times 10^{-15}}{(1.6022 \times 10^{-19})\lambda_A} = \frac{12398}{\lambda_A}.$$

(b)   To examine $d\lambda$, take the derivative of $\lambda = hc/E$

$$d\lambda = -\frac{hc}{E^2}dE = -\left(\frac{hc}{E}\right)\frac{dE}{E} = -\lambda\frac{dE}{E}.$$

Notice that dE/E is dimensionless if both dE and E are given in the same units. Demand that both dE and E are specified in eV. From part (a), then

$$\lambda = -\lambda\frac{dE_{eV}}{E_{eV}} = -\frac{\lambda\lambda_A}{12398}dE_{eV}.$$

This would give $d\lambda$ in meters when λ is given in meters (note that the two factors of λ appearing here are in different units: meters and Å, respectively.) But, converting both $d\lambda$ and to Å would introduce factors of 10^{10} on both sides, which cancel each other out. Hence

$$d\lambda_A = -\frac{\lambda_A^2}{12398}dE_{eV}.$$

For $\lambda_A = 4000$ and dE = 0.01, $d\lambda_A = 12.91$Å.

Problem 1-18

Compute the de Broglie wavelength for an electron of energy 1 keV. If this electron were interacting with a hydrogen atom, would classical mechanics suffice?

From the de Broglie relation,

$$\lambda = \frac{h}{\sqrt{2Km}} = \frac{6.626 \times 10^{-34} \ \text{J} - \text{sec}}{\sqrt{2(1.602 \times 10^{-16} \ \text{J}) \ (9.11 \times 10^{-31} \ \text{kg})}} = 3.88 \times 10^{-11} \ \text{m} = 0.39 \text{Å}.$$

This result is on the order of the size of the hydrogen atom, therefore quantum mechanics must be used.

Problem 1-19

Particles in motion have associated de Broglie wavelengths. As a rule, if one is within a few de Broglie wavelengths of a particle then the particle loses its identity as a particle and acts more like a wave, with the result that classical mechanics no longer suffices to describe collisions. Rutherford investigated the scattering of α-particles (helium nuclei) from stationary gold nuclei, using classical mechanics to analyze the collisions. This question investigates the issue from a wave-mechanics perspective. (a) Determine the de Broglie wavelength of an α-particle of kinetic energy 6 million electron-volts. (b) The electrical potential energy of charges Q_1 and Q_2 separated by distance r is given by $U = Q_1 Q_2 / 4\pi\varepsilon_0 r$. By equating the initial kinetic energy of the α-particle to the electrical potential energy, find the distance of closest approach of the α-particle to the gold nucleus, and so render a judgment as to how well-justified Rutherford was in his use of classical mechanics.

The de Broglie wavelength of a 6-MeV α-particle (m = 6.64×10^{-27} kg) is

$$\lambda = \frac{h}{\sqrt{2Km}} = \frac{6.626 \times 10^{-34} \text{J} - \text{sec}}{\sqrt{2(6)(1.602 \times 10^{-13} \text{J}/\text{MeV})(6.64 \times 10^{-27} \text{kg})}} = 5.87 \times 10^{-15} \ \text{m}.$$

Gold has Z $-$ 79; the closest-approach distance is then

$$K = \frac{(2e)(79e)}{4\pi\varepsilon_0 r} \quad \rightarrow \quad r = \frac{158e^2}{4\pi\varepsilon_0 K} = 3.79 \times 10^{-14} \ \text{m}.$$

The ratio of the closest-approach distance to λ is about 6.47: not terribly large. Rutherford's use of classical mechanics was marginally justified.

Problem 1-20

In kinetic theory, atoms are modeled as point masses of mass m, and the mean speed v of an atom in an environment at absolute temperature T is given by $mv^2 = 3kT$ where k is Boltzmann's constant. Derive an expression for the de Broglie wavelength for an atom of mass m at absolute temperature T in terms of h, k, T, and m. At room temperature, hydrogen atoms typically travel many thousands of Ångstroms between collisions; is it fair to treat them as effectively acting as point masses under such circumstances?

With $K = mv^2/2 = 3kT/2$, the de Broglie wavelength becomes

$$\lambda = \frac{h}{\sqrt{2Km}} = \frac{h}{\sqrt{3mkT}}$$

For a hydrogen atom at T = 300K, say, and using MKS units,

$$\lambda = \frac{h}{\sqrt{3mkT}} = \frac{6.626 \times 10^{-34}}{\sqrt{3(1.67 \times 10^{-27})(300)(1.38 \times 10^{-23})}} = 1.46 \times 10^{-10} \text{ m},$$

or about 1.46Å. It is safe to regard the hydrogen atoms as point masses.

Problem 1-22

An atom of mass m is initially at rest with respect to an outside observer and has an electron in an excited state. The electron transits to a lower energy level, emitting a single photon of wavelength λ in the process. The emitted photon will have momentum p = h/λ. To conserve momentum, the atom must recoil with some speed v. Derive an expression giving the ratio of the photon's energy to the kinetic energy of the recoiling atom in terms of λ, m, c, and h. A classical analysis will suffice for the recoiling atom. Evaluate the ratio numerically for a hydrogen atom emitting a photon of wavelength 5000Å. Based on your result, do you think it was reasonable to neglect the energy involved with the recoiling atom in the derivation of the Bohr model? Along these lines, the momentum transferred to an atom upon its absorbing a photon has actually been measured: see Physics Today 2005;58(7):9.

The energy of the photon is $E = hc/\lambda$. From equation (1.4.1), the momentum of the photon is $p = h/\lambda$. Hence, for the recoiling atom, $mv = h/\lambda$, so $v = h/\lambda m$. The ratio is

$$\frac{E_{photon}}{E_{recoil}} = \frac{hc}{\lambda}\left(\frac{2}{mv^2}\right) = \frac{hc}{\lambda}\left(\frac{2}{m}\right)\left(\frac{\lambda m}{h}\right)^2 = \frac{2mc\lambda}{h}.$$

For a hydrogen atom and a photon of λ = 5000Å,

$$\frac{E_{photon}}{E_{recoil}} = \frac{2(1.67 \times 10^{-27})(2.998 \times 10^{8})(5 \times 10^{-7})}{(6.626 \times 10^{-34})} = 7.56 \times 10^{8}.$$

The vast majority of the transition energy is carried off by the photon. It is safe to neglect the recoiling atom in the derivation of the Bohr model.