Final 2011

Note Title 12/21/2011

1.
$$KE = \frac{1}{2}mv^2 = \frac{3}{2}kT$$
$$= \frac{p^2}{2m}$$
$$\Rightarrow p = \sqrt{3mkT}$$
$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{3mkT}}$$

2.
$$H\psi(x) = E\psi(x)$$
$$\Rightarrow \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right) A\exp(-x^2)$$
$$= EA\exp(-x^2) \quad - \text{(*)}$$

$$\frac{d}{dx} e^{-x^2} = -2x e^{-x^2}$$
$$\frac{d^2}{dx^2} e^{-x^2} = \frac{d}{dx}\left(\frac{d}{dx} e^{-x^2}\right) = \frac{d}{dx}\left(-2x e^{-x^2}\right)$$
$$= -2\left[e^{-x^2} + x\frac{d}{dx}e^{-x^2}\right]$$
$$= -2\left[e^{-x^2} - 2x^2 e^{-x^2}\right]$$
$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right) e^{-x^2}$$
$$= -\frac{\hbar^2}{2m}(-2)\left(e^{-x^2} - 2x^2 e^{-x^2}\right) + V(x) e^{-x^2}$$
$$= \left[\frac{\hbar^2}{m}(1-2x^2) + V(x)\right] e^{-x^2}$$

Thus

$$(*) \Rightarrow \left[V(x) + \frac{\hbar^2}{m}(1-2x^2) - E \right] e^{-x^2} = 0$$

$$\Rightarrow V(x) = E - \frac{\hbar^2}{m} + \frac{2\hbar^2}{m} x^2$$

3. $\Delta x \Delta p \geq \frac{\hbar}{2} \Rightarrow \Delta p \geq \frac{\hbar}{2\Delta x} \approx \frac{\hbar}{2L}$

$$E = \frac{p^2}{2m} = \frac{\Delta p^2}{2m} \geq \frac{1}{2m}\left(\frac{\hbar}{2L}\right)^2$$

$$\geq \frac{\hbar^2}{8mL^2}$$

4. $A(\theta,\varphi) = C = \text{const} \cdot Y_{0,0}(\theta,\varphi)$

$$= Y_{0,0}(\theta,\varphi)$$

$$L^2 Y_{0,0} = 0\, Y_{0,0}$$

$$L_z Y_{0,0} = 0\, Y_{0,0}$$

Thus $\langle L^2 \rangle = \langle L_z \rangle = 0$

5. (a) Both electrons occupy ψ_{100} state

So the spins should be in the singlet state in order to make the total wave-function antisymmetric

total spin quantum number $\underline{\underline{0}}$

spatial wave function symmetric

(b) total spins can be either in the singlet state ($S=0$) or in the triplet state ($S=1$)

(c) With electron-electron interaction, parallel spins ($S=1$) have the lower energy configuration, because electrons tend to stay away from each other.

The spatial wave function should be antisymmetric

(d) Electron-electron interaction is repulsive and thus contributes positive energy

$$\therefore \quad E_b < E_c$$

6. (a)

$$1 = \int_0^{2\pi} |\psi(x)|^2 dx$$

$$= A^2 \int_0^{2\pi} (1-e^{-ix})(1-e^{ix})\, dx$$

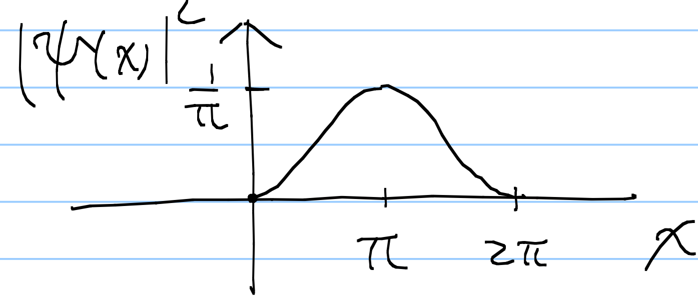
$$= A^2 \int_0^{2\pi} (1 - e^{-ix} - e^{ix} + 1)\, dx$$

$$= A^2 \cdot 4\pi \quad \Rightarrow A = \frac{1}{\sqrt{4\pi}}$$

(b) $$|\psi(x)|^2 = \frac{1}{4\pi}\left(2 - e^{-ix} - e^{ix}\right)$$

$$= \frac{1}{4\pi}\left(2 - 2\cos x\right)$$

$$= \begin{cases} \frac{1}{2\pi}(1-\cos x) & \text{for } x \in [0, 2\pi] \\ 0 & \text{else} \end{cases}$$



(c) $$\langle H \rangle = \frac{1}{4\pi}\int_0^{2\pi} (1-e^{-ix})\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)(1-e^{ix})\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi}(1-e^{-ix})\left(-\frac{\hbar^2}{2m}\right)\left(-i^2 e^{ix}\right)dx$$

$$= -\frac{\hbar^2}{8\pi m}\int_0^{2\pi}(1-e^{-ix})\,e^{ix}\,dx$$

$$= -\frac{\hbar^2}{8\pi m}\int_0^{2\pi}(e^{ix}-1)\,dx$$

$$= \frac{\hbar^2}{8\pi m}\cdot 2\pi = \underline{\frac{\hbar^2}{4m}}$$

(d) $$P(E_2) = |a_2|^2$$

$$a_2 = \int_0^{2\pi} \psi_2^*(x)\,\psi(x)\,dx$$

$$= \sqrt{\frac{2}{2\pi}} \int_0^{2\pi} \sin\left(\frac{2\pi}{2\pi} x\right) \cdot \frac{1}{\sqrt{4\pi}} (1 - e^{ix})\, dx$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \sin(x)(1 - e^{ix})\, dx$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} \sin(x)\, e^{ix}\, dx$$

$$= -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{ix} - e^{-ix}}{2i} e^{ix}\, dx$$

$$= -\frac{1}{4\pi i} \int_0^{2\pi} (e^{2ix} - 1)\, dx$$

$$= -\frac{1}{4\pi i} \cdot (-2\pi)\, dx$$

$$= \frac{1}{2i}\, dx$$

$$\Rightarrow \quad P(E_2) = |a_2|^2 = \underline{\frac{1}{4}}$$

7.

$$T_0 = \exp\left(-2 \int_0^{a} \frac{\sqrt{2m(V_0 - E)}}{\hbar}\, dx\right)$$

$$= \exp\left(-2 \frac{\sqrt{2m(V_0 - E)}}{\hbar}\, a\right)$$

$$T = \exp\left(-2 \int_0^{2a} \frac{\sqrt{2m(V_0 - E)}}{\hbar}\, dx\right)$$

$$= \exp\left(-4 \frac{\sqrt{2m(V_0 - E)}}{\hbar}\, a\right)$$

$$= T_0^2$$

8. (a) $1 = A^2(4 + 1 + 9)$

$$\Rightarrow A = \frac{1}{\sqrt{14}}$$

(b) $\langle L^2 \rangle = \hbar^2 \left(\frac{4}{14} 0 + \frac{1}{14} 1 \cdot 2 + \frac{9}{14} \cdot 2 \cdot 3 \right)$

$$= \hbar^2 \frac{2+54}{14} = \hbar^2 \frac{56}{14} = 4\hbar^2$$

(c) $\langle L_z \rangle = \hbar \left(\frac{4}{14} 0 + \frac{1}{14} 0 + \frac{9}{14} \cdot (-2) \right)$

$$= -\frac{9}{7} \hbar$$

(d) $\psi_{new} = cL_-(2\psi_{100} - i\psi_{210} + 3\psi_{32-2})$

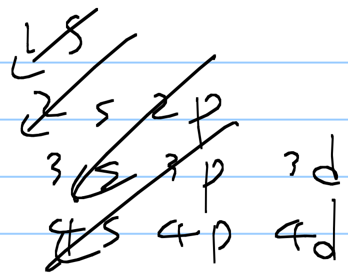
$$= c(0 - \text{const}\,\psi_{21-1} + 0)$$

$$= \psi_{21-1}$$

$$\langle L^2 \rangle_{\psi_{21-1}} = \hbar^2 1 \cdot 2 = 2\hbar^2$$

$$\langle L_z \rangle_{\psi_{21-1}} = -\hbar$$

9.



$1s^2\ 2s^2\ 2p^6\ 3s^2\ 3p^4$

$$\begin{array}{cccc} & -1 & 0 & 1 \\ p & \boxed{\uparrow} & \boxed{\uparrow} & \boxed{\uparrow\downarrow} \end{array}$$

$S = \frac{1}{2} + \frac{1}{2} = 1$

$L = -1 \times 1 + 0 \times 1 + 1 \times 2 = 1$

$J = L + S = 2$
(more than half-filled)

$2S+1 = 3$

$\therefore\ {}^3P_2$

10. $2\sqrt{2m} \int_{x_1}^{x_2} \sqrt{E - V(x)}\, dx = nh$

$\Rightarrow 4\sqrt{2m} \int_0^{x_{turn}} \sqrt{E - \alpha x}\, dx = nh$

$E - \alpha x_{turn} = 0$

$\Rightarrow x_{turn} = \frac{E}{\alpha}$

$\Rightarrow 4\sqrt{2m} \int_0^{\frac{E}{\alpha}} \sqrt{E - \alpha x}\, dx = nh$

$$\alpha x \equiv y$$

$$\Rightarrow 4\sqrt{2m} \int_0^E \sqrt{E-y}\,\frac{dy}{\alpha} = nh$$

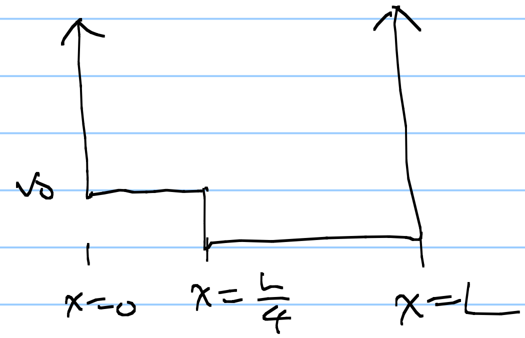
$$\Rightarrow \frac{4\sqrt{2m}}{\alpha}\left[-\frac{2}{3}(E-y)^{\frac{3}{2}}\right]_0^E = nh$$

$$\Rightarrow \frac{8\sqrt{2m}}{3\alpha}\cdot E^{\frac{3}{2}} = nh$$

$$\Rightarrow E^{\frac{3}{2}} = \frac{3\alpha nh}{8\sqrt{2m}}$$

$$\Rightarrow \underbrace{E = \left(\frac{3\alpha nh}{8\sqrt{2m}}\right)^{\frac{2}{3}}}, \quad n=1,2,\cdots$$

11.



$$E_1 = E_1^0 + \varepsilon_1 = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 + \varepsilon_1$$

$$\varepsilon_1 = \langle \psi_1 | \Delta V | \psi_1 \rangle$$

$$= \frac{2}{L}\int_0^{\frac{L}{4}} \sin\left(\frac{\pi}{L}x\right) V_0 \sin\left(\frac{\pi}{L}x\right) dx$$

$$= \frac{2V_0}{L}\int_0^{\frac{L}{4}} \sin^2\left(\frac{\pi}{L}x\right) dx$$

$$= \frac{2V_0}{L}\left[\frac{x}{2} - \frac{1}{4}\cdot\frac{L}{\pi}\sin\left(\frac{2\pi}{L}x\right)\right]_0^{\frac{L}{4}}$$

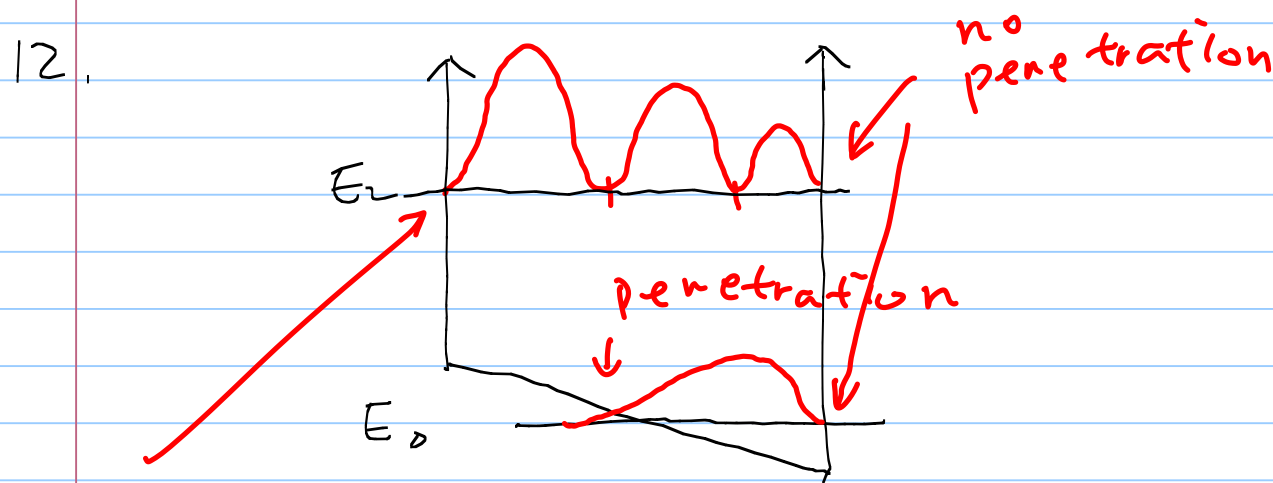
$$= \frac{2V_0}{L}\left[\frac{L}{8} - \frac{L}{4\pi}\sin\left(\frac{2\pi}{L}\frac{L}{4}\right)\right]$$

$$= 2V_0\left[\frac{1}{8} - \frac{1}{4\pi}\right]$$

$$= V_0\left[\frac{1}{4} - \frac{1}{2\pi}\right]$$

$$\therefore\ E_1 = \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2 + V_0\left(\frac{1}{4} - \frac{1}{2\pi}\right)$$

12.



→ KE increases (p increases)
λ decreases

amplitude decreases
(lower probability of
observing the particle)