Lecture 9

The risk - neutral formulation

Provides better understanding of Black - Scholes pricing formula. (option)

Consider $\mu(t) = \mu$, $\sigma(t) = \sigma$, then

$$dS(t) = \mu S(t)\, dt + \sigma S(t)\, dW(t)$$

is solved by

$$S(T) = S(t)\, e^{(\mu - \frac{\sigma^2}{2})(T-t) + \sigma[W(T) - W(t)]}.$$

In particular,

$$\langle S(T) \rangle \Big|_{s,t} = s\, e^{\mu(T-t)}.$$

On average, just regular increase or decrease in value

Now note that $\underbrace{H[S(T)]}_{\text{value of the option @ time } T} = \begin{cases} 0, & S(T) < K \\ S(T) - K, & S(T) \geq K \end{cases}$

Then $\langle H[S(T)] \rangle \Big|_{s,t} = \int_K^\infty dS\, (S-K)\, P_\mu(S,T|s,t) =$

$K = e^M$, $\tilde{y} = \log S$; $y = \log s$, s = initial price @ t

$$= \int_M^\infty d\tilde{y}\, (e^{\tilde{y}} - e^M)\, p_\mu(\tilde{y}, T | y, t),$$

where

$$p_\mu(\tilde{y}, T | y, t) = \frac{1}{\sqrt{2\pi(T-t)}}\, e^{-\frac{[\tilde{y} - y - (\mu - \frac{\sigma^2}{2})(T-t)]^2}{2\sigma^2(T-t)}}$$

gaussian

Now, it would be reasonable to set

$$F(s,t) = \underbrace{e^{\mu(t-T)}}_{\substack{\text{interest} \\ \text{rate factor}}} \underbrace{\langle H[S(T)] \rangle \Big|_{s,t}}_{\text{expected gain @ time } T}$$

Same as Black-Scholes formula, but with μ instead of r.

If we set $\mu = r$ (or some r_{eff} computed as an average of many μ's), we behave risk-neutrally and recover the Black-Scholes valuation. In this case, our prediction for stocks is risk-neutral as well:

$$S(T) = S(t)\, e^{\overset{\mu \to r}{\uparrow}(r - \frac{\sigma^2}{2})(T-t) + \sigma[W(T) - W(t)]}.$$

Girsanov's theorem

Two probability measures are equivalent if $P(A) > 0 \iff Q(A) > 0$.

A = any set of events

Events possible under P are also possible under Q, although the probabilities of these events are in general different under the two models.

Now, consider an SDE

Itô $\quad dx(t) = f(t)dt + dW(t)$, or

$$\Delta x_i = f_i \Delta t_i + \Delta W_i .$$

Prob. of W n steps later:

$$P(W) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \Delta t_i}} e^{-\frac{\Delta W_i^2}{2\Delta t_i}} .$$

If we define $\Delta V_i = \Delta W_i + f_i \Delta t_i$, we get

$$\tilde{P}(V) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \Delta t_i}} e^{-\frac{(\Delta V_i - f_i \Delta t_i)^2}{2\Delta t_i}} =$$

$$= e^{\sum_i [f_i \Delta V_i - \frac{1}{2} f_i^2 \Delta t_i]} \underbrace{\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \Delta t_i}} e^{-\frac{\Delta V_i^2}{2\Delta t_i}}}_{P(V)} \;\; \text{⊜}$$

$$\underset{\substack{n\to\infty \\ \text{limit}}}{\text{⊜}} \underbrace{e^{\int_0^T f(t)dV(t) - \frac{1}{2}\int_0^T dt\, f^2(t)}}_{\text{always positive pre-factor}} P(V) .$$

initial time

Thus, $\tilde{P}(V)$ & $P(V)$ are equivalent in the sense of P & Q above: any path that is possible under $\tilde{P}(V)$ is also possible under $P(V)$, but they are weighted differently due to the pre-factor.

Our SDE becomes:

$$dx(t) = dV(t)$$, where

$V(t)$ is assigned the measure $\tilde{P}(V)$.

More generally, two SDEs are equivalent if $b(t) = g(t)$:

$$\begin{cases} dx(t) = f(t)\,dt + b(t)\,dW(t), \\ dy(t) = a(t)\,dt + g(t)\,dW(t). \end{cases}$$

Girsanov's theorem

Thus, we can write

$$dy(t) = a(t)dt + b(t)dV(t),$$ where

$V(t)$ is assigned the measure

$$\tilde{P}(V) = e^{\int_0^T \varphi(t)\,dV(t) - \frac{1}{2}\int_0^T dt\, \varphi^2(t)} \underbrace{P(V)}_{\text{Wiener measure}},$$

where $\varphi(t) = \dfrac{f(t) - a(t)}{b(t)}$.

The previous example had $a(t) = 0$,

$b(t) = g(t) = 1$, s.t. $\varphi(t) = f(t)$.

Conversely, if $b(t) \neq g(t)$, the two stochastic processes are never equivalent.

Lévy processes

Homogeneous in probability space variables. Recall the differential CK equation in 1D, with $A(z,t)=0$ and $B(z,t)=0$ for simplicity:

$$\frac{\partial p(z,t|y,t')}{\partial t} = \int dx \left[W(z|x,t)\, p(x,t|y,t') - W(x|z,t)\, p(z,t|y,t') \right]$$

Consider $W(z|x,t) \to \underbrace{w(z-x)}_{= w(x-z)}$, isotropic.

homogeneous, no explicit t-dependence

Then $p(z,t|\overset{=0}{y},t') = p(z-\underset{=0}{y}, t|t') \equiv p(z,t)$.

$$\frac{\partial p(z,t)}{\partial t} = \int_{-\infty}^{\infty} dx \left[w(\underbrace{z-x}_{=u}) p(x,t) - \overbrace{w(x-z)}^{= w(z-x)} p(z,t) \right] \ominus$$

$u \Rightarrow x = z-u$

$$\ominus \int_{-\infty}^{\infty} du\, w(u) \left[p(z-u,t) - p(z,t) \right].$$

Define characteristic function:

$$\varphi(s,t) = \int_{-\infty}^{\infty} dz\, e^{isz} p(z,t), \text{ then}$$

$$\frac{\partial \varphi(s,t)}{\partial t} = \int_{-\infty}^{\infty} du\, w(u) \int_{-\infty}^{\infty} dz\, e^{isz} p(z-u,t) \diamond$$

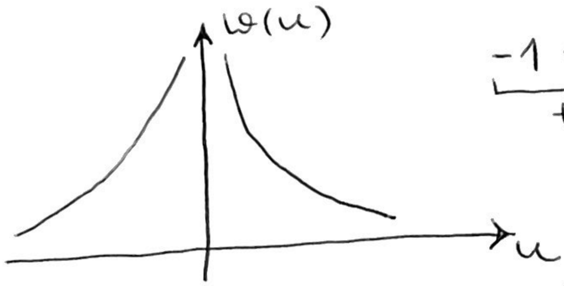
$\tilde{z} = z - u$

$$\ominus \int_{-\infty}^{\infty} du\, w(u) \underbrace{\int_{-\infty}^{\infty} dz\, e^{isz} p(z,t)}_{\varphi(s,t)} =$$

$$= \int_{-\infty}^{\infty} du\, w(u) \left[e^{isu} - 1 \right] \times \varphi(s,t) \quad , \text{ yielding}$$

$$\boxed{\varphi(s,t) = e^{\int_{-\infty}^{\infty} du\, (e^{isu} - 1)\, w(u) \times t}} \; .$$

Now, consider $w(u) \sim \dfrac{1}{|u|^{\alpha+1}}$,



$\underbrace{-1 < \alpha}_{\text{to have divergence @ } u=0} < 2$

Since $\int^{\infty} du\, w(u) \sim u^{-\alpha}$, the divergence is integrable for $\alpha < 0$

If $\alpha \leq 1$,

$$\int_0^{\infty} du\, w(u)\, (e^{isu} - 1) \sim \int_0^{\infty} du\, \underbrace{\frac{1}{u^{\alpha+1}}}_{u^{-\alpha}}\, u \Rightarrow \text{finite around } u=0 \text{ if } \alpha \leq 1$$

regularization not needed

[$\alpha = 1$ gives log divergence]

For $1 < \alpha < 2$, the integral is divergent around $u = 0$: regularization necessary

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Now, postulate asymmetric $w(u)$:

$$w(u) = \begin{cases} \dfrac{A}{|u|^{\alpha+1}} \;, & u < 0 \\[2ex] \dfrac{B}{u^{\alpha+1}} \;, & u > 0 \end{cases}$$

$0 < \alpha < 2$ [For $\alpha > 2$, divergences cannot be regularized]

For $1<\alpha<2$, the principal value integral is defined as:

$$\fint_{-\infty}^{\infty} du\, w(u)\left(e^{isu}-1\right) \equiv \lim_{\epsilon\to 0}\left\{\int_{-\infty}^{-\delta(\epsilon)} du\, w(u)\left(e^{isu}-1\right)+\right.$$

$$\left.+\int_{\epsilon}^{\infty} du\, w(u)\left(e^{isu}-1\right)\right\}.$$

<u>If $A=B$,</u> $\delta(\epsilon)=\epsilon$ and we obtain:

$$\int_{\epsilon}^{\infty} du\, u^{-\alpha} \sim u^{-\alpha+1}\Big|_{\epsilon}^{\infty} \Rightarrow -\epsilon^{1-\alpha} \text{ at the lower limit}$$

$$\uparrow$$

$$e^{isu}-1 \approx isu$$

$$\int_{-\infty}^{-\epsilon} du\,(-u)^{-\alpha-1}\, u = \int_{-\infty}^{-\epsilon} d(\underbrace{-u}_{=w})\,(-u)^{-\alpha} =$$

$$= \int_{\infty}^{\epsilon} dw\, w^{-\alpha} \sim w^{-\alpha+1}\Big|_{\infty}^{\epsilon} \Rightarrow \epsilon^{1-\alpha} \text{ at the upper limit,}$$

cancels the divergence

<u>If $A\neq B$,</u> $\delta(\epsilon)=\epsilon\left(\frac{B}{A}\right)^{\frac{1}{1-\alpha}}$, yielding:

$$\overset{B}{\cancel{A}}\int_{\epsilon}^{\infty} du\, u^{-\alpha} \Rightarrow -\overset{B}{\cancel{A}}\,\epsilon^{1-\alpha}$$

$$\overset{A}{\cancel{B}}\int_{-\infty}^{-\delta(\epsilon)} du\,(-u)^{-\alpha-1}\, u \sim \overset{A}{\cancel{B}}\, w^{-\alpha+1}\Big|_{\infty}^{\epsilon\left(\frac{B}{A}\right)^{\frac{1}{1-\alpha}}} \Rightarrow$$

$$\Rightarrow \overset{A}{\cancel{B}}\,\epsilon^{1-\alpha}\left[\left(\frac{B}{A}\right)^{\frac{1}{1-\alpha}}\right]^{1-\alpha} = \overset{B}{\cancel{A}}\,\epsilon^{1-\alpha},$$

divergences cancel again

Now, consider $\underline{u>0}$:

$$B\int_0^\infty du\, u^{-\alpha-1} e^{isu} = B\left(\frac{i}{s}\right)\int_0^\infty dt\, e^{-t}\left(\frac{i}{s}\right)^{-\alpha-1} t^{-\alpha-1} =$$

$$isu=-t\,,\quad u=i\frac{t}{s}$$

$$= B\left(\frac{i}{s}\right)^{-\alpha}\underbrace{\int_0^\infty dt\, e^{-t}\, t^{-\alpha-1}}_{``\Gamma(-\alpha)"} \quad \left[\Gamma(z)=\int_0^\infty dt\, t^{z-1}e^{-t}\right] =$$

$$\begin{cases} \underset{\sim\sim}{s>0}: & s=|s|\,, \quad i^{-\alpha}=\left(e^{i\frac{\pi}{2}}\right)^{-\alpha}=\cos\left(\frac{\pi}{2}\alpha\right)-i\sin\left(\frac{\pi}{2}\alpha\right), \\ \underset{\sim\sim}{s<0}: & s=-|s|\,, \quad (-i)^{-\alpha}=\left(e^{-i\frac{\pi}{2}}\right)^{-\alpha}=\cos\left(\frac{\pi}{2}\alpha\right)+i\sin\left(\frac{\pi}{2}\alpha\right), \end{cases}$$

$$= B|s|^{\alpha}\Gamma(-\alpha)\left[\cos\frac{\pi\alpha}{2}-i\frac{s}{|s|}\sin\frac{\pi\alpha}{2}\right].$$

Note that $\int_0^\infty du\, u^{-\alpha-1} = \lim_{s\to 0}\int_0^\infty du\, u^{-\alpha-1}\overline{\overline{e^{isu}}} \to 0$.

Likewise, for $\underline{u<0}$:

$$A\int_{-\infty}^0 du\,(-u)^{-\alpha-1}e^{isu} = A\left(-\frac{i}{s}\right)\int_{-\infty}^0 dt\, e^{t}\left(\frac{it}{s}\right)^{-\alpha-1} =$$

$$isu=t\,,\quad u=-i\frac{t}{s}$$

$$= A\left(-\frac{i}{s}\right)\left(\frac{i}{s}\right)^{-\alpha-1}(-1)^{\alpha+1}\underbrace{\int_{-\infty}^0 dt\, e^{t}(-t)^{-\alpha-1}}_{w=-t} =$$

$$\int_{-\infty}^0 dt = \int_0^\infty dw$$

$$= A\left(-\frac{i}{s}\right)^{-\alpha}\Gamma(-\alpha) = A|s|^{\alpha}\Gamma(-\alpha)\left[\cos\frac{\pi\alpha}{2}+i\frac{s}{|s|}\sin\frac{\pi\alpha}{2}\right]$$

$$\begin{cases} \underset{\sim\sim}{s>0}: & (-i)^{-\alpha}=\left(e^{-i\frac{\pi}{2}}\right)^{-\alpha}=\cos\frac{\pi\alpha}{2}+i\sin\frac{\pi\alpha}{2} \\ \underset{\sim\sim}{s<0}: & i^{-\alpha}=\cos\frac{\pi\alpha}{2}-i\sin\frac{\pi\alpha}{2} \end{cases}$$

$\int_{-\infty}^0 du\,(-u)^{-\alpha-1} \to 0$, as with $u<0$ above

Putting it all together, we obtain:

$$\varphi(s,t) = e^{t|s|^\alpha \Gamma(-\alpha)\left[(A+B)\cos\frac{\pi\alpha}{2} + i\frac{s}{|s|}(A-B)\sin\frac{\pi\alpha}{2}\right]}.$$

Define
$$\begin{cases} \gamma = -(A+B)\Gamma(-\alpha)\cos\frac{\pi\alpha}{2}, \\ \beta = \dfrac{A-B}{A+B}. \end{cases}$$

characteristic function of Paretian process

Then $\varphi(s,t) = e^{-t|s|^\alpha \gamma\left[1 + i\beta\frac{s}{|s|}\tan\frac{\pi\alpha}{2}\right]} \equiv$

$$\equiv \int_{-\infty}^{\infty} du\, e^{isu} \underbrace{Par(u|\alpha,\beta,\gamma t)}_{\text{Paretian prob. distribution}}.$$

If we set $\alpha = 2$ (despite earlier restriction to $\alpha < 2$), and assume $\gamma = \frac{1}{2}$, $\beta = 0$ $\leftarrow A=B$, we obtain $\varphi(s,t) = e^{-t\frac{s^2}{2}}$, the charact. f'n of the Wiener process ($\mu = 0$, $\sigma^2 = t$).

However, $\gamma \sim \Gamma(-2)\underbrace{\cos\pi}_{-1} = \infty$, so we have replaced ∞ by $\frac{1}{2}$, or assumed $A \to 0$. (!)

If we set $\alpha = 1$ and $\beta = 0$ $(A = B)$, we get

$\Gamma(-\alpha)\cos\frac{\pi\alpha}{2} \to -\frac{\pi}{2}$, yielding

$$\varphi(s,t) = e^{-\frac{\pi}{2} t|s| \times \underbrace{2A}_{\frac{\gamma}{\pi/2} = \frac{2\gamma}{\pi}}} = \underbrace{e^{-\gamma t|s|}}_{\text{charact. function of the Cauchy process}}$$

We focus now on the $\alpha \to 1$ limit, which requires separate treatment.

Here, we use $A \log \delta(\epsilon) = B \log \epsilon$ in the principal value integral. $\rightarrow \delta(\epsilon) = \epsilon^{B/A}$

Indeed, $B \int_{\epsilon}^{\infty} du \frac{1}{u} = B \log u \Big|_{\epsilon}^{\infty} \Rightarrow -B \log \epsilon$

$$-A \int_{-\infty}^{-\delta(\epsilon)} du\, (-u)^{-1} \underset{w=-u}{=} A \int_{\infty}^{\delta(\epsilon)} dw\, \frac{1}{w} = A \log w \Big|_{\infty}^{\epsilon^{\frac{B}{A}}} \Rightarrow$$

$$\Rightarrow A \frac{B}{A} \log \epsilon = \underline{\underline{B \log \epsilon}}.$$

divergences cancel

Next, we add the drift term to the equation for $p(z,t)$:

$$\frac{\partial p(z,t)}{\partial t} = -\underset{\text{const}}{a} \frac{\partial p(z,t)}{\partial z} + \int_{-\infty}^{\infty} du\, w(u) [p(z-u,t) - p(z,t)].$$

Then
$$-a \int_{-\infty}^{\infty} dz\, e^{isz} \frac{\partial p}{\partial z} \overset{\text{by parts}}{=} a \int_{-\infty}^{\infty} dz\, (is)\, e^{isz} p =$$

$$= \underline{\underline{is a\, \varphi(s,t)}}.$$

Thus,

$$\frac{\partial \varphi(s,t)}{\partial t} = \left[ias + \int_{-\infty}^{\infty} du \left(e^{isu} - 1\right) w(u) \right] \times \varphi(s,t),$$

yielding

$$\varphi(s,t) = e^{\left\{ \underbrace{ias}_{\text{additional term}} + \int_{-\infty}^{\infty} du \left(e^{isu} - 1\right) w(u) \right\} t}.$$

Next, consider

$$\underbrace{B \int_0^\infty du\, u^{-2} e^{isu}}_{-\int_0^\infty du \left(\frac{d}{du} u^{-1}\right) e^{isu}} \xRightarrow[\text{leave } B \text{ out for now}]{\text{by parts, throw out } \infty \text{ boundary term}} is \int_0^\infty du\, \underbrace{u^{-1}}_{\frac{d}{du}\log u} e^{isu} \Rightarrow$$

$$\Rightarrow -(is)^2 \int_0^\infty du \log u\, e^{\overset{isu}{\underset{-t}{=}}} \Leftrightarrow$$

by parts, throw out boundary term

$\log(0) = -\infty$

$$u = -\frac{t}{is} = \frac{i}{s} t$$

$$\int_0^\infty du \Rightarrow \frac{i}{s} \int_0^\infty dt$$

$$\Leftrightarrow s^2 \frac{i}{s} \int_0^\infty dt\, e^{-t} \log\left(\frac{i}{s} t\right) = is \left[\log\left(\frac{i}{s}\right) \underbrace{\int_0^\infty dt\, e^{-t}}_{=1} + \underbrace{\int_0^\infty dt\, e^{-t} \log t}_{= -\gamma \simeq 0.577} \right] = is \left[\log\left(\frac{i}{s}\right) - \gamma \right].$$

Euler's constant

$\underline{s>0}$: $i|s|\big[\underbrace{\log i}_{=\log(\underbrace{e^{i\frac{\pi}{2}}}_{=i}) = i\frac{\pi}{2}} - \log|s| - \gamma\big] = -|s|\frac{\pi}{2} - i|s|\{\log|s| + \gamma\}.$

$\underline{s<0}$: $-i|s|\big[\underbrace{\log(-i)}_{-i\frac{\pi}{2}} - \log|s| - \gamma\big] =$

$= -|s|\frac{\pi}{2} + i|s|\{\log|s| + \gamma\}.$

Note that $\lim_{s\to 0}\int_0^\infty du\, u^{-2} e^{isu} = 0$, after regularization

Likewise,

$$A\int_{-\infty}^{0} du\,(-u)^{-2} e^{isu} \underset{\text{omit A for now}}{\Rightarrow} \int_0^\infty d\tilde{u}\,\tilde{u}^{-2} e^{-is\tilde{u}} \Rightarrow \qquad \tilde{u} = -u$$

$$\Rightarrow -is\left[\log\left(-\frac{i}{s}\right) - \gamma\right].$$

$\underline{s>0}$: $-i|s|\big[\underbrace{\log(-i)}_{=-i\frac{\pi}{2}} - \log|s| - \gamma\big] = -|s|\frac{\pi}{2} + i|s|\{\log|s| + \gamma\}.$

$\underline{s<0}$: $i|s|\big[\underbrace{\log i}_{i\frac{\pi}{2}} - \log|s| - \gamma\big] = -|s|\frac{\pi}{2} - i|s|\{\log|s| + \gamma\}.$

$\lim_{s\to 0}\int_{-\infty}^{0} du\,(-u)^{-2} e^{isu} = 0$, after regularization

Putting it all together, we obtain:

$$\varphi(s,t) = e^{-|s|t\left\{\frac{\pi}{2}(A+B) \mp \frac{is}{|s|}\left[+a + (A-B)(\log|s| + \gamma)\right]\right\}}$$

Define $\begin{cases} \gamma = \dfrac{\pi}{2}(A+B), \\ \beta = \dfrac{A-B}{A+B}. \end{cases}$

Choose $\alpha = (B-A)\gamma$, then

$$\underbrace{\varphi(s,t)}_{\text{charact. f'n for Paretian process, } \alpha=1} = e^{-|s|t\gamma\left\{1-\beta\frac{is}{|s|}\frac{2}{\pi}\log|s|\right\}} \equiv$$

becomes Cauchy process if $\beta = 0$

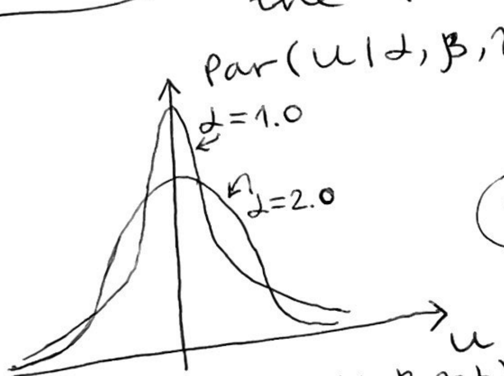
$$\equiv \int_{-\infty}^{\infty} du\, e^{isu}\, \underbrace{\mathrm{Par\,I}(u|\beta,\gamma t)}_{\text{Paretian prob. distr'n, } \alpha=1}.$$

charact. f'n for Paretian process, $\alpha=1$

$1<\alpha<2$: all moments higher than 1^{st} diverge, 1^{st} moment $=0$ when for $\beta\neq 0$ (asymmetric jumps)

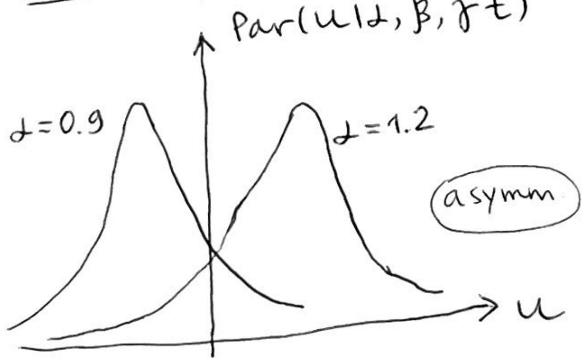
$0<\alpha\leq 1$: all moments diverge, including the 1^{st} moment

$\beta=0$,
$\gamma t=1$



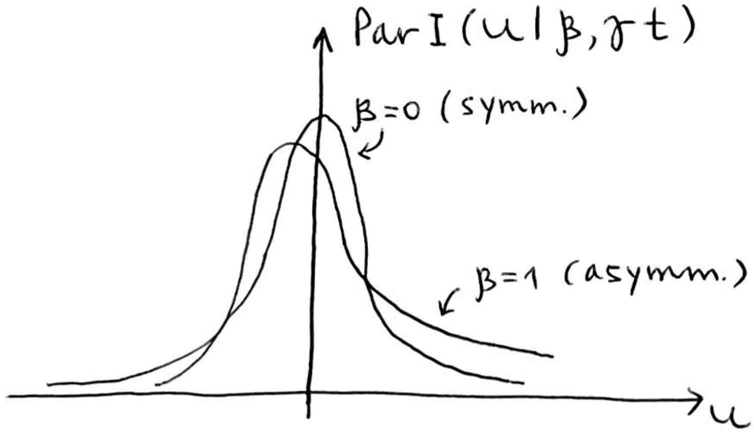
symm.

$\beta=0.3$,
$\gamma t=1$



asymm

mode can increase or decrease as $\alpha\uparrow$



$\alpha = 1$,
$\gamma t = 1$