# Lecture 8

## [ Brownian motion description of financial markets ]

Neglects large jumps in stock or commodity prices (observed in data, leading to heavy-tailed distrib's) $\Rightarrow$ still a fruitful paradigm which allows us to use SDE machinery.

<u>assets</u>: (i) Bonds ($\approx$ savings account, except one can buy & sell them, return rate $r$)

$$dB(t) = r B(t) dt$$

$B(t)$ = value of a bond @ time $t$

(ii) Stocks (securities traded on the stock market) $S(t)$ = value of one share @ time $t$

$$dS(t) = \underbrace{\mu(t)}_{\text{drift}} S(t) dt + \underbrace{\sigma(t)}_{\text{volatility}} S(t) dW(t)$$

(iii) Derivatives We'll focus on options: the right to purchase (a "call" option) or sell (a "put" option) a certain amount of stock at a fixed price $K$ per share, at some future time $T$

One can sell assets without owning them (hoping to acquire them in the future, before the actual sale) $\Rightarrow$

$\Rightarrow$ "selling the assets short", or "taking a short position on the asset".

---

Actually owning assets is "taking a long position on the asset".

Thus, both positive and negative quantities of assets are admissible.

---

Perfect liquidity assumption: all assets are traded freely and traders can acquire any ⊕ or ⊖ amount of any asset.

## The Black-Scholes Formula

Fundamental question: develop a model to predict option prices.

Def. $F(\underbrace{S(t)}_{\text{current value of stock}}, \underbrace{t}_{\text{time}})$ = price of an option to buy one unit of stock

$$dF(S(t),t) = \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}dS + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}dS^2 =$$

$$= \left\{ \frac{\partial F}{\partial t} + \mu(t)S(t)\frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2(t)S^2(t)\frac{\partial^2 F}{\partial S^2} \right\} dt +$$

$$+ \underbrace{\sigma(t)S(t)\frac{\partial F}{\partial S}dW(t)}_{\text{noise term}}$$

Is it possible to construct a portfolio of stocks, options and bonds s.t. all the fluctuations cancel?

---

| Portfolio | Value |
|---|---|
| → One option (held short) | $-F(S(t),t)$ |
| → $\Delta(t)$ shares of stock (held long) | $\Delta(t)S(t)$ |
| → $\beta(t)$ bonds (held long) | $\beta(t)B(t)$ |

Total value $P(t) = -F(S(t),t) + \Delta(t)S(t) +$
$+ \beta(t)B(t)$.

$$dP(t) = -dF + d\{\Delta \times S + \beta \times B\}.$$

Impose the self-financing condition: closed system, no net inflow or outflow of capital.

Consider discretized dynamics:

$$\begin{cases} S(t) \to S_n & S(t) + dS(t) \to S_{n+1} \\ B(t) \to B_n & B(t) + dB(t) \to B_{n+1} \\ \Delta(t) \to \Delta_n & \Delta(t) + d\Delta(t) \to \Delta_{n+1} \\ \beta(t) \to \beta_n & \beta(t) + d\beta(t) \to \beta_{n+1} \end{cases}$$

$n$ $\qquad$ $n+1$

We require that $(\Delta_{n+1} - \Delta_n) S_{n+1} + (\beta_{n+1} - \beta_n) B_{n+1} = 0.$

self-financing condition

use 'n+1' prices

Then $d\Delta(t)\left[S(t) + dS(t)\right] + d\beta(t)\left[B(t) + dB(t)\right] = 0$ $\quad (*)$

Now, consider $d\{\Delta \times S + \beta \times B\} =$

$= \underline{d\Delta \times S} + \underline{d\Delta \times dS} + \Delta dS + \underline{d\beta \times B} + \underline{d\beta \times dB} + \beta dB =$

$= \Delta dS + \beta dB.$

In Eq. $(*)$, $d\beta(t)\, dB(t) = 0$ since $dB(t) \sim dt$ [no stochastic component]

Then $d\beta(t) = -(S(t) + dS(t)) \dfrac{d\Delta(t)}{B(t)}$

SDE for $\beta(t)$ if $S(t), \Delta(t), B(t)$ are known

Now, recall that

$$dP(t) = -\left\{ \frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right\} dt -$$

$$- \sigma S \frac{\partial F}{\partial S} dW + \Delta \left\{ \mu S dt + \sigma S \, dW \right\} +$$

$$+ \beta \times r B dt =$$

choose $\Delta = \frac{\partial F}{\partial S}$ to cancel the $dW$ terms

$$= -\left\{ \frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - r\beta B \right\} dt .$$

$dP(t)$ has no noise term $\Rightarrow$ $P(t)$ does not fluctuate. Thus, $P(t)$ behaves like a bond $\Rightarrow$ $dP(t) = \tilde{r} P(t) dt$.

The no arbitrage, or "financial equilibrium" condition states that $\tilde{r} = r$. If $\tilde{r} < r$, it's easier to just buy bonds. If $\tilde{r} > r$, one can borrow bonds and make money at the rate $\tilde{r} - r$.

Thus, $rP = -\frac{\partial F}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + r\beta B$, or

$$-rF + r\frac{\partial F}{\partial S} S + \cancel{r\beta B} = -\frac{\partial F}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \cancel{r\beta B} .$$

Finally,

$$\underbrace{\frac{\partial F(s,t)}{\partial t} = rF(s,t) - rs\,\frac{\partial F(s,t)}{\partial s} - \frac{1}{2}\sigma^2(t)\,s^2\,\frac{\partial^2 F}{\partial s^2}}_{\text{Black-Scholes equation}} \qquad (**)$$

Consider $\sigma(t) = \sigma$ for simplicity, and define the backward Kolmogorov equation:

$$\frac{\partial P(x,T\,|\,s,t)}{\partial t} = -rs\,\frac{\partial P(x,T\,|\,s,t)}{\partial s} - \frac{\sigma^2 s^2}{2}\,\frac{\partial^2 P(x,T|s,t)}{\partial s^2}, \qquad (***)$$

$$P(x,T\,|\,s,T) = \delta(x-s) \quad \text{final condition.}$$

Then $F(s,t)$ in Eq. $(**)$ is given by:

$$F(s,t) = e^{r(t-T)}\int dx\, P(x,T\,|\,s,t)\,F(x,T).$$

Indeed, $\displaystyle \frac{\partial F(s,t)}{\partial t} = rF_{(s,t)} + e^{r(t-T)}\int dx \left\{ -rs\,\frac{\partial P}{\partial s} - \frac{\sigma^2 s^2}{2}\,\frac{\partial^2 P}{\partial s^2} \right\} F(x,T) = rF(s,t) +$

$$+ e^{r(t-T)}(-rs)\,\frac{\partial}{\partial s}\left[\int dx\, P\, F(x,T)\right] +$$

$$+ e^{r(t-T)}\left(-\frac{\sigma^2 s^2}{2}\right)\frac{\partial^2}{\partial s^2}\left[\int dx\, P\, F(x,T)\right] =$$

$$= rF(s,t) - rs\,\frac{\partial F(s,t)}{\partial s} - \frac{\sigma^2 s^2}{2}\,\frac{\partial^2 F(s,t)}{\partial s^2}.$$

Eq. (***) is equivalent to the following SDE:

$$dx(t) = x(t)\left[r\,dt + \sigma\, dW(t)\right]$$

Use $y = \log x$: $\begin{cases} a = rx, \\ b = \sigma x. \end{cases}$

$$dy = \left[rx\frac{1}{x} + \frac{\sigma^2 x^2}{2}\left(-\frac{1}{x^2}\right)\right]dt + \sigma x \frac{1}{x}\, dW(t) =$$

$$= \left(r - \frac{\sigma^2}{2}\right)dt + \sigma\, dW(t).$$

Then $y(T) = y(t) + \left(r - \frac{\sigma^2}{2}\right)(T-t) +$

$$+\sigma\left[W(T) - W(t)\right].$$

Thus, $y(T)$ is Gaussian-distributed:

$$p(\bar{y}, T \mid y, t) = \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \times$$

$$\times e^{-\frac{\left[\bar{y} - y - \left(r - \frac{\sigma^2}{2}\right)(T-t)\right]^2}{2\sigma^2(T-t)}}.$$

Finally,

$$F(s,t) = e^{r(t-T)} \int d\bar{y}\; p(\bar{y}, T \mid y, t)\, F(e^{\bar{y}}, T). \quad (+)$$

Final condition: $F(s,T) = \begin{cases} 0, & s < K \\ s - K, & s \geq K \end{cases}$

Recall that $s$ = stock price at time T

$K$ = 'strike price' (stock price written on the option)

Then Eq. (+) becomes:

$$F(s,t) = e^{r(t-T)} \int_M^\infty d\bar{y}\,(e^{\bar{y}} - e^M)\, p(\bar{y},T|y,t)$$

define $K = e^M$

Introducing $\tilde{N}(z) = \frac{1}{\sqrt{2\pi}}\int_z^\infty d\tilde{z}\, e^{-\frac{\tilde{z}^2}{2}}$, we obtain for the 2nd term:

$$\frac{1}{\sqrt{2\pi\sigma^2(T-t)}}\int_M^\infty e^{-\frac{[\bar{y}-\overbrace{(y+(r-\frac{\sigma^2}{2})(T-t))}^{\equiv\alpha}]^2}{2\sigma^2(T-t)}}\, d\bar{y} =$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}}\int_{\frac{\log K-\alpha}{\sigma\sqrt{T-t}}}^\infty dx\, e^{-\frac{x^2}{2}}} = \tilde{N}\left(\frac{\log K - \overbrace{\log S}^{\text{"}y} - (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right).$$

$$x = \frac{\bar{y}-\alpha}{\sigma\sqrt{T-t}} \Rightarrow dx = \frac{d\bar{y}}{\sigma\sqrt{T-t}} \; ; \; \int_M^\infty d\bar{y} \Rightarrow \int_{\frac{\log K-\alpha}{\sigma\sqrt{T-t}}}^\infty dx \times \sigma\sqrt{T-t}$$

---

[$x \Leftrightarrow \tilde{z}$ here]

Introduce $\tilde{x} = -x$:

$$d\tilde{x} = -dx \; ; \; \int_z^\infty dx \Rightarrow -\int_{-z}^{-\infty} d\tilde{x} = \int_{-\infty}^{-z} d\tilde{x}.$$

Then $\tilde{N}(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-z} d\tilde{x}\, e^{-\frac{\tilde{x}^2}{2}} \equiv N(-z)$,

where $N(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} dx\, e^{-\frac{x^2}{2}}$ is the cumulative gaussian distribution

---

Thus, the entire 2nd term is given by

$$K e^{r(t-T)}\, N\left(\frac{\log\frac{S}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}\right).$$

Likewise, one can derive the 1st term, resulting in

$$F(s,t) = S\,N\Bigg(\overbrace{\frac{\log\frac{S}{K} + (r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}}^{\equiv d_1}\Bigg) -$$

$$- K e^{r(t-T)} N\Bigg(\underbrace{\frac{\log\frac{S}{K} + (r-\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}}_{\equiv d_2}\Bigg)$$

Black-Scholes option pricing formula

Consider the $T \to t$ limit:

if $S > K$, $\log\frac{S}{K} > 0$ and $d_1 \to +\infty$, $d_2 \to +\infty$.

Then $F(s,T) = S\underbrace{N(+\infty)}_{=1} - K\,N(+\infty) = S - K$, as expected

Likewise, if $S < K \Rightarrow$ $d_1 \to -\infty$, and $d_2 \to -\infty$

$F(s,T) = S\underbrace{N(-\infty)}_{=0} - K\,N(-\infty) = 0$, again as expected

Consider the $T \to \infty$ limit:

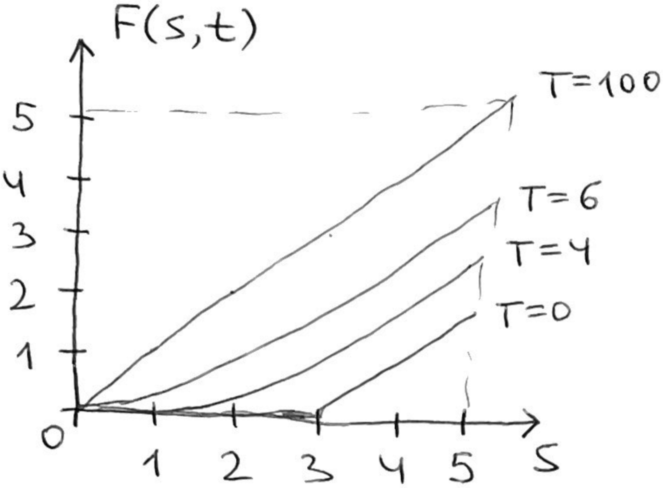
$d_1 \to +\infty$, $d_2 \to \begin{cases} +\infty, & r > \frac{\sigma^2}{2} \\ -\infty, & r < \frac{\sigma^2}{2} \end{cases}$

[if $r = \frac{\sigma^2}{2}$, $d_2 \to 0$]

Regardless of $d_2$, $F(s,t) = S\,N(+\infty) = S$

Option price @ $t$ is equal to the stock price @ $t$ : $S(t)$

Finally, note that $F(s,t) \to 0$ as $S \to 0$, for any $T-t$

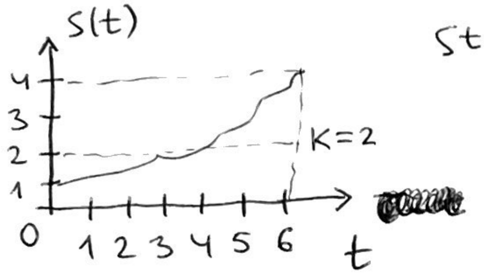



$t=0$,
$K=3$,
$r=0.2$, $(0.02?)$
$\sigma=0.2$

initial stock price ↙ initial quantity of bonds ↙

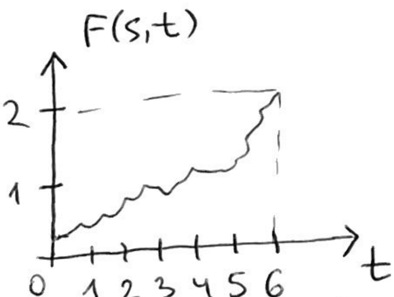
**Ex. 1** $S(0)=1$, $B(0)=1$, $\beta(0)=10$

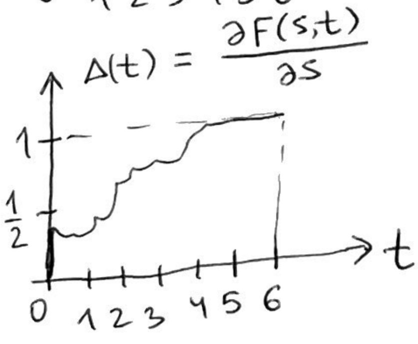
$K=2$ $\sigma, r, \mu$ also given



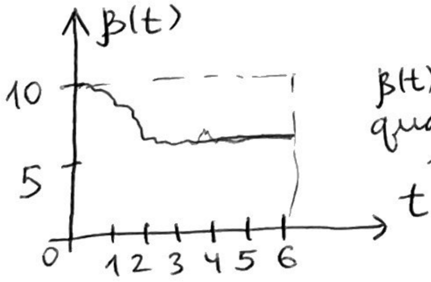

Strong growth starting from $S(0)=1$

time to maturity : $T-t$
as $t\uparrow$, it's as if $T\downarrow$ in the F(s,t) vs. s plot above
$T=6$ in the plots on the left

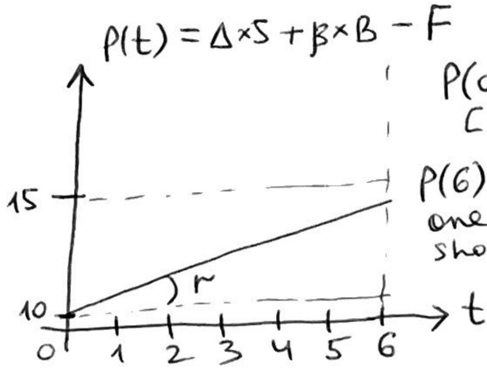



$\Delta(t) \to 1$ eventually
(portfolio contains one unit of stock)




$\beta(t) \to$ fixed quantity of bonds




$P(0)=10$
[all bonds]

$P(6)=$ one unit of stock, one option (held short), and $<10$ bonds