

Lecture 3A

Backward & Forward Kolmogorov Equations

FP eq'n:

$$\frac{\partial p(x,t)}{\partial t} = - \frac{\partial}{\partial x} \left[\underbrace{\mu(x,t)}_{\frac{\langle \Delta x \rangle}{\Delta t}} p(x,t) \right] + \frac{1}{2} \frac{\partial^2}{\partial x^2} \left[\underbrace{\sigma^2(x,t)}_{\frac{\langle \Delta x^2 \rangle}{\Delta t}} p(x,t) \right] + \mathcal{O}(\Delta t)$$

Prob. density: $p(x,t | \underbrace{x_0, t_0}_{\text{IC}}) = p(x,t)$

Now, consider

~~etc.~~

Prob. (not density):

$$P(A, t | x_0, t_0) = P(X_t \in A | X_{t_0} = x_0)$$

$$= \int_A dy p(y, t | x_0, t_0)$$

$$\text{In particular, } P(X_t \leq x | X_{t_0} = x_0) \equiv \equiv P(x, t | x_0, t_0) =$$

$$= \int_{-\infty}^x p(z, t | x_0, t_0) dz$$

$$\rightarrow \frac{\partial P(x, t | x_0, t_0)}{\partial x} = p(x, t | x_0, t_0)$$

Now, use CK:
$$p(x, t | x_0, t_0) = \int dy p(x, t | y, t_1) \times p(y, t_1 | x_0, t_0)$$

Consider

$$\frac{p(x, t | x_0, t_0 - \Delta t) - p(x, t | x_0, t_0)}{\Delta t} \quad (\text{with } \Delta t \text{ small}) \quad (\equiv)$$

$$p(x, t | x_0, t_0 - \Delta t) = \int dy p(x, t | y, t_0) p(y, t_0 | x_0, t_0 - \Delta t)$$

$$\begin{aligned} (\equiv) \frac{1}{\Delta t} & \left[\int dy p(x, t | y, t_0) p(y, t_0 | x_0, t_0 - \Delta t) - \right. \\ & \left. - p(x, t | x_0, t_0) \underbrace{\int dy p(y, t_0 | x_0, t_0 - \Delta t)}_{=1} \right] = \end{aligned}$$

$$= \frac{1}{\Delta t} \int dy [p(x, t | y, t_0) - p(x, t | x_0, t_0)] \times p(y, t_0 | x_0, t_0 - \Delta t) \approx$$

$$\begin{aligned} \approx & \frac{\partial p}{\partial x_0} \underbrace{\frac{1}{\Delta t} \int dy (y - x_0) p(y, t_0 | x_0, t_0 - \Delta t)}_{\mu(x_0, t_0)} + \\ & + \frac{1}{2} \frac{\partial^2 p}{\partial x_0^2} \underbrace{\frac{1}{\Delta t} \int dy (y - x_0)^2 p(y, t_0 | x_0, t_0 - \Delta t)}_{\sigma^2(x_0, t_0)} \end{aligned}$$

Finally,

$$-\frac{\partial P(x, t | x_0, t_0)}{\partial t_0} = \mu(x_0, t_0) \frac{\partial P}{\partial x_0} + \frac{1}{2} \sigma^2(x_0, t_0) \frac{\partial^2 P}{\partial x_0^2} \quad (*)$$

apply $\frac{\partial}{\partial x}$ to Eq. (*):

$$-\frac{\partial p(x, t | x_0, t_0)}{\partial t_0} = \mu(x_0, t_0) \frac{\partial p}{\partial x_0} + \frac{1}{2} \sigma^2(x_0, t_0) \frac{\partial^2 p}{\partial x_0^2}$$

eq'n for prob. density

Connection with FPT:

$$P_v(x_0, t-t_0) = \int dy p(y, t | x_0, t_0) \quad \diamond$$

homog.

survival probability
(no absorption) @ time t

$$\diamond 1 - \underbrace{\varphi(x_0, t-t_0)}_{\text{prob. that FPT is } < t-t_0}$$

Then Eq. (*) gives: [apply $\int dy \times \dots$]

$$-\frac{\partial \varphi(x_0, t-t_0)}{\partial t_0} = \mu(x_0, t_0) \frac{\partial \varphi}{\partial x_0} + \frac{1}{2} \sigma^2(x_0, t_0) \frac{\partial^2 \varphi}{\partial x_0^2}$$

Finally, $\frac{\partial}{\partial t_0} f(t-t_0) = -\frac{\partial}{\partial t} f(t-t_0)$,
yielding
 $\left(\begin{array}{l} t_0 \rightarrow 0, \\ x_0 \rightarrow x \end{array} \right)$

$$\frac{\partial \mathcal{G}(x,t)}{\partial t} = \mu(x) \frac{\partial \mathcal{G}(x,t)}{\partial x} + \frac{\epsilon^2(x)}{2} \frac{\partial^2 \mathcal{G}(x,t)}{\partial x^2}.$$

QED