HW #3 (2025)

① (a) Prove that $\int_{t_0}^{t} G(t')\,[dW(t')]^{2+N} = 0$

for $N = 1, 2, 3, \ldots$, where $G(t)$ is an arbitrary non-anticipating function.

(b) Similarly, prove that

$$\int_{t_0}^{t} G(t')\,dt'\,dW(t') = 0 .$$

② Consider a 1D Brownian particle moving in a constant gravitational field:

Itô

$$dx = -g\,dt + \sqrt{D}\,dW(t) .$$

The particle moves on the interval $[a, b]$ with reflecting boundary conditions.

Find the steady-state solution $p_{ss}(x)$ of the corresponding Fokker-Planck equation.

(3.) Consider 1D Ornstein-Uhlenbeck process:

$$dx = -kx\,dt + \sqrt{D}\,dW(t) \,. \qquad \text{Itô}$$

Simulate this SDE using the Cauchy-Euler approach. Plot $\tilde{p}_{ss}(x)$ extracted from the simulations and compare it with

$$p_{ss}(x) = \sqrt{\frac{k}{\pi D}}\, e^{-\frac{kx^2}{D}}$$ derived in class.