Maxwell's fridge

$T_h > T_c$

$\left[\begin{matrix}\text{coupled} \\ \text{to } T_h\end{matrix}\right] \begin{cases} — u \\ — d \end{cases}$ $\qquad \Delta E = E_u - E_d > 0$

$\rightarrow$ [|0|1|0|1|0|] ... $\rightarrow$

T_c

$$\frac{R_{d\to u}}{R_{u\to d}} = e^{-\overset{(k_B T_h)^{-1}}{\overset{\shortparallel}{\beta_h}} \Delta E} \quad \Big] \text{ intrinsic to demon}$$

As discussed previously,

$$\begin{cases} R_{d\to u} = \gamma(1-\sigma), \\ R_{u\to d} = \gamma(1+\sigma), \end{cases} \qquad \sigma = \tanh\frac{\beta_h \Delta E}{2}.$$

$$0 < \sigma < 1$$

Bit states can change only through bit-demon interactions, of duration $\underline{\underline{\tau}}$: $0d \Leftrightarrow 1u$. These transitions are coupled to T_c:

$$\frac{R_{0d\to 1u}}{R_{1u\to 0d}} = e^{-\underbrace{\beta_c}_{(k_B T_c)^{-1}} \Delta E}$$

$$\begin{cases} R_{0d\to 1u} = 1-\omega, \\ R_{1u\to 0d} = 1+\omega, \end{cases} \qquad \omega = \tanh\frac{\beta_c \Delta E}{2}$$

$$0 < \omega < 1$$

Note that
$$\epsilon \equiv \frac{\omega - \sigma}{1-\omega\sigma} = \frac{\frac{e^{\beta_c\Delta E}-1}{e^{\beta_c\Delta E}+1} - \frac{e^{\beta_h\Delta E}-1}{e^{\beta_h\Delta E}+1}}{1 - \frac{(e^{\beta_c\Delta E}-1)}{(e^{\beta_c\Delta E}+1)}\frac{(e^{\beta_h\Delta E}-1)}{(e^{\beta_h\Delta E}+1)}} =$$

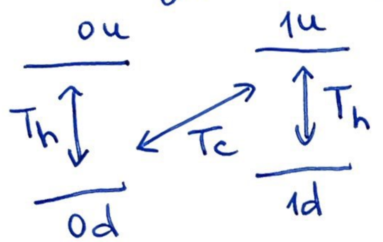
$$= \frac{(e^{\beta_c\Delta E}-1)(e^{\beta_h\Delta E}+1) - (e^{\beta_h\Delta E}-1)(e^{\beta_c\Delta E}+1)}{(e^{\beta_c\Delta E}+1)(e^{\beta_h\Delta E}+1) - (e^{\beta_c\Delta E}-1)(e^{\beta_h\Delta E}-1)} \boxed{=}$$

$$\boxed{\epsilon}\ \frac{e^{\beta_c \Delta E} - e^{\beta_h \Delta E} - e^{\beta_h \Delta E} + e^{\beta_c \Delta E}}{e^{\beta_c \Delta E} + e^{\beta_h \Delta E} + e^{\beta_c \Delta E} + e^{\beta_h \Delta E}} =$$

$$= \frac{e^{\beta_c \Delta E} - e^{\beta_h \Delta E}}{e^{\beta_c \Delta E} + e^{\beta_h \Delta E}} = \tanh \frac{(\beta_c - \beta_h)\Delta E}{2}.$$

Note that $0 < \epsilon < 1$

Finally, as before we have $\delta = p_0 - p_1$:



4 states

Assume that each incoming bit is in state $\emptyset$! $\delta = 1$.

Then the system is in $(\emptyset u, \emptyset d)$ states to begin with. After τ, if we end up in $(\emptyset u, \emptyset d)$ again: all $0d \to 1u$ transitions were balanced by $1u \to 0d$ transitions $\Rightarrow$
$\Rightarrow$ no energy exchanged with the T_c reservoir.
If, however, we end up in $(1u, 1d) \Rightarrow \Delta E$ withdrawn from the cold reservoir.
<u>Thermal rectification</u>: energy withdrawn from T_c but never delivered to it.
In the end, flux of energy $T_c \to T_h$.
The energy exchange events imprinted on the <u>outgoing</u> bit stream.

More generally: $\begin{cases} \delta > 0 \text{ (excess of } \emptyset\text{'s)} : T_c \to T_h \text{ flow} \\ \delta < 0 \text{ (excess of 1's)} : T_h \to T_c \text{ flow} \end{cases}$

At ss, $\varphi = p_1' - p_1 = \dfrac{\delta - \delta'}{2}$

Average energy transfer per interaction interval : $Q_{c \to h} = \varphi \Delta E$.

If $Q_{c \to h} > 0 \Rightarrow$ acts like a demon

Now, define

$$S(\delta) = -\sum_{i=0,1} p_i \log p_i = -\frac{1-\delta}{2} \log \frac{1-\delta}{2} - \frac{1+\delta}{2} \log \frac{1+\delta}{2}.$$

↑ information content per bit

$$\Delta S_B \equiv S(\delta') - S(\delta) = \underbrace{S(\delta - 2\varphi) - S(\delta)}.$$

One can show that

$$\varphi(\underbrace{\delta, \epsilon, \gamma, \omega, \tau}_{\equiv \Lambda}) = \frac{\delta - \epsilon}{2} \underbrace{\eta(\Lambda)}_{>0}$$

As before,

$$T_{2\times 2} = \overbrace{\underbrace{\begin{matrix} 1010 \\ 0101 \end{matrix}}} e^{R\tau} \begin{pmatrix} p_0 & 0 \\ 0 & p_0 \\ p_1 & 0 \\ 0 & p_1 \end{pmatrix} \quad \Bigg\} \quad \underset{\text{demon}}{\begin{pmatrix} p_u \\ p_d \end{pmatrix}} \quad \underset{\text{bit}}{\begin{pmatrix} p_0 \\ p_1 \end{pmatrix}}$$

$$\tau \to \emptyset: \; T_{2\times 2} \begin{pmatrix} p_u \\ p_d \end{pmatrix} = \overbrace{\underbrace{\begin{matrix} 1010 \\ 0101 \end{matrix}}} \underbrace{\begin{pmatrix} p_0 & 0 \\ 0 & p_0 \\ p_1 & 0 \\ 0 & p_1 \end{pmatrix} \begin{pmatrix} p_u \\ p_d \end{pmatrix}}_{\begin{pmatrix} p_u p_0 \\ p_d p_0 \\ p_u p_1 \\ p_d p_1 \end{pmatrix}} = \begin{pmatrix} p_u \\ p_d \end{pmatrix},$$

as expected

$$R = \begin{array}{c} \begin{array}{cccc} \emptyset u & \emptyset d & 1u & 1d \end{array} \\ \begin{pmatrix} \times & \gamma(1-\sigma) & 0 & 0 \\ \gamma(1+\sigma) & \times & 1+\vartheta & 0 \\ 0 & 1-\vartheta & \times & \gamma(1-\sigma) \\ 0 & 0 & \gamma(1+\sigma) & \times \end{pmatrix} \end{array} \Rightarrow e^{R\tau}$$

$$\underset{2\times 2}{T} \begin{pmatrix} p_u \\ p_d \end{pmatrix}_{ss} = \begin{pmatrix} p_u \\ p_d \end{pmatrix}_{ss}$$

$$\begin{pmatrix} p_0' \\ p_1' \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} e^{R\tau} \begin{pmatrix} p_u^{ss} p_0 \\ p_d^{ss} p_0 \\ p_u^{ss} p_1 \\ p_d^{ss} p_1 \end{pmatrix} \Rightarrow \text{get } \varphi.$$

$\varphi = \frac{\delta - \epsilon}{2} \underbrace{\eta(\lambda)}_{\text{can be shown} > 0}$ in general,

$\varphi \to \frac{\delta-\epsilon}{2}\left[1 - e^{-(1-\sigma\vartheta)\tau}\right]$ as $\gamma \to \infty$.

So, $\text{sign}(\varphi) = \text{sign}(\delta - \epsilon)$.

$\delta > \epsilon \Rightarrow \varphi > 0 \Rightarrow Q_{c\to h} = \varphi \Delta E > 0$.

A surplus of $\emptyset$'s prevails over the T difference: flow of energy $T_c \to T_h$.

Now consider

$$\frac{d\vec{p}}{dt} = R\vec{p}.$$

as $\tau \uparrow$, the system relaxes to <u>SS</u>:

$R\vec{p}_{ss} = 0$, where

$$\begin{pmatrix} -\gamma(1+\sigma) & \gamma(1-\sigma) & 0 & 0 \\ \gamma(1+\sigma) & (\vartheta-1)-\gamma(1-\sigma) & 1+\vartheta & 0 \\ 0 & 1-\vartheta & -(1+\vartheta)-\gamma(1+\sigma) & \gamma(1-\sigma) \\ 0 & 0 & \gamma(1+\sigma) & -\gamma(1-\sigma) \end{pmatrix} \begin{pmatrix} 1 \\ \mu \\ \mu\nu \\ \mu^2\nu \end{pmatrix} \; (=)$$

$$\mu = \frac{1+\sigma}{1-\sigma}, \quad \nu = \frac{1-\vartheta}{1+\vartheta}$$

$$(=) \begin{pmatrix} -\gamma(1+\sigma) + \mu\gamma(1-\sigma) \\ \gamma(1+\sigma) + \mu(\vartheta-1) - \mu\gamma(1-\sigma) + \mu\nu(1+\vartheta) \\ \mu(1-\vartheta) - \mu\nu(1+\vartheta) - \mu\nu\gamma(1+\sigma) + \mu^2\nu\gamma(1-\sigma) \\ \mu\nu\gamma(1+\sigma) - \mu^2\nu\gamma(1-\sigma) \end{pmatrix} =$$

$$= \begin{pmatrix} -\gamma(1+\sigma) + \gamma(1+\sigma) \\ \gamma(1+\sigma) + \frac{1+\sigma}{1-\sigma}(\vartheta-1) - \gamma(1+\sigma) + \frac{1+\sigma}{1-\sigma}(1-\vartheta) \\ \frac{1+\sigma}{1-\sigma}(1-\vartheta) - \frac{1+\sigma}{1-\sigma}(1-\vartheta) - \mu\nu\gamma(1+\sigma) + \mu\nu\gamma(1+\sigma) \\ \mu\nu\gamma(1+\sigma) - \mu\nu\gamma(1+\sigma) \end{pmatrix} =$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \text{ as desired.}$$

With norm'n:

$$\vec{p}_{ss} = \begin{pmatrix} 1 \\ \mu \\ \mu\nu \\ \mu^2\nu \end{pmatrix} \frac{1}{(1+\mu)(1+\mu\nu)}$$

Note that $p_{ss,ij} = p^D_{i,ss}\, p^B_{j,ss}$

$$\begin{cases} \vec{p}^{\,D}_{ss} = \frac{1}{1+\mu}\begin{pmatrix} 1 \\ \mu \end{pmatrix} \\ \vec{p}^{\,B}_{ss} = \frac{1}{1+\mu\nu}\begin{pmatrix} 1 \\ \mu\nu \end{pmatrix} \end{cases}$$

Define KL distance:

$$D(\vec{p}\,|\,\vec{p}_{ss}) = \sum_{k\in(\emptyset u, \emptyset d, 1u, 1d)} p_k \log \frac{p_k}{p_{k,ss}} \geq 0$$

as $\vec{p}$ approaches $\vec{p}_{ss}$, $D \downarrow$:

$$\frac{d}{dt} D(\vec{p}\,|\,\vec{p}_{ss}) \leq 0.$$

Define $\begin{cases} \vec{p}_0 \\ \vec{p}_\tau \end{cases}$ ← demon-bit distr's @ $t=0$ & τ

$$D(\vec{p}_0|\vec{p}_{ss}) - D(\vec{p}_\tau|\vec{p}_{ss}) \geq 0 \quad (*)$$

Then $\underbrace{\sum_k p_{0,k}\log p_{0,k} - \sum_k p_{\tau,k}\log p_{\tau,k}}_{S_\tau - S_0} - \Big[\sum_k p_{0,k}\log p_{ss,k} - \sum_k p_{\tau,k}\log p_{ss,k}\Big] =$

$$= S_\tau - S_0 + \sum_k (p_{\tau,k} - p_{0,k})\log \underbrace{p_{k,ss}}_{\text{factorizes}} =$$

$$= S_\tau - S_0 + \underbrace{\sum_{i\in\{u,d\}}}_{\text{demon states}} (p^D_{\tau,i} - p^D_{0,i})\log p^D_{i,ss} +$$

$$+ \sum_{j\in\{0,1\}} (p^B_{\tau,j} - p^B_{0,j})\log p^B_{j,ss}$$

The joint entropy of the demon-bit system: $S' = S'^D + S^B - \underbrace{I(D;B)}_{\geq 0}$ mutual info, decreases the entropy for correlated systems

$S' = -\sum_i \underbrace{p_i(D,B)}_{\text{joint prob.}}\log p_i(D,B)$ $=$ $p(D,B) = p(D|B)\,p(B)$ ~~$p(D)p(B)$~~

$= -\sum_i p_i(D|B)\,p_i(B)\,[\log p_i(B) + \log p_i(D|B)]$

$S(X,Y) = S(X) + S(Y) - I(X;Y)$

$I_0(D;B) = 0$

$I_\tau(D;B)$ may be >0 (demon-bit correlations)

$I(X;Y) = \sum_{x,y} p(x,y)\log\frac{p(x,y)}{p(x)p(y)}$

$S^D_\tau = S^D_0 \Leftarrow$ demon in SS

So, $S_\tau - S_0 = \Delta S_B - I_\tau(D;B)$

Also, $\vec{p}_0^{\,D} = \vec{p}_\tau^{\,D}$ in SS :

(*) gives

$$\Delta S_B - I_\tau(D;B) + \sum_{j\in\{0,1\}} (p_{\tau,j}^B - p_{0,j}^B) \log p_{j,ss}^B \geq 0$$

At SS, $p_{\tau,0}^B - p_{0,0}^B = -(p_{\tau,1}^B - p_{0,1}^B) = -\varphi$

So, $\sum_{j\in\{0,1\}} (p_{\tau,j}^B - p_{0,j}^B) \log p_{j,ss}^B =$

$$= -\varphi \log \frac{1}{1+\mu\nu} + \varphi \log \frac{\mu\nu}{1+\mu\nu} =$$

$$= \varphi \log \mu\nu = \varphi \log \frac{(1+\sigma)(1-\omega)}{(1-\sigma)(1+\omega)} =$$

$$= \varphi \log \frac{R_{0u\to d}}{R_{d\to u}} + \varphi \log \frac{R_{0d\to 1u}}{R_{1u\to 0d}} =$$

$$= \varphi \beta_h \Delta E - \varphi \beta_c \Delta E = (\beta_h - \beta_c) \underbrace{\varphi \Delta E}_{Q_{c\to h}}$$

So, $\Delta S_B + Q_{c\to h}(\beta_h - \beta_c) \geq I_\tau(D;B) > 0$

strong "Clausius statement"

Thus if $\varphi > 0 \Rightarrow Q_{c\to h} > 0$, $\left(\Rightarrow Q_{c\to h}(\beta_h - \beta_c) \leq 0\right)$

we obtain: $\Delta S_B > 0$

"information is written to the bit stream"

Now if $\epsilon > \delta > 0 \Rightarrow \varphi < 0 \Rightarrow Q_{c\to h} = \underbrace{\varphi}_{<0}\underbrace{\Delta E}_{>0} < 0$,

energy flows from $T_h \to T_c$.

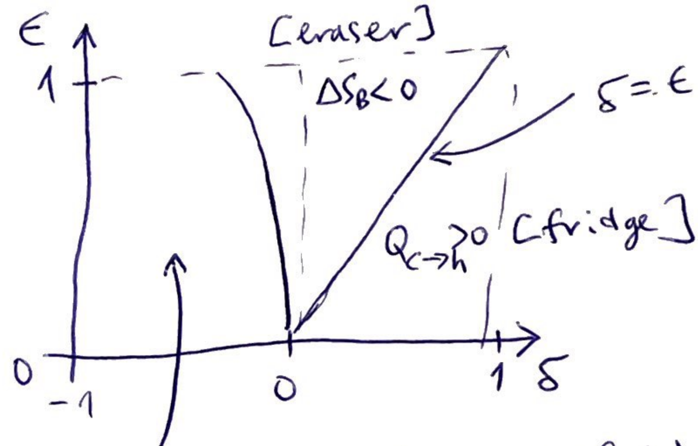
$$Q_{c\to h}(\beta_h - \beta_c) > 0 \quad \Rightarrow \quad \Delta S_B < 0$$

$$\text{s.t. } Q_{c\to h}(\beta_h - \beta_c) + \Delta S_B \geq 0$$

$\Delta S_B < 0$ here b/c $\underbrace{\delta' > \delta}_{\text{b/c } \varphi<0 \text{ "}\frac{\delta-\delta'}{2}\text{"}} > 0$

So, information is erased from the bit stream.

Fix ω, γ, τ



$\begin{cases} Q_{c\to h} < 0 \text{ (hot} \to \text{cold)} \\ \Delta S_B > 0 \end{cases}$

"both thermodynamic & information resources are consumed"