# Final Exam Solutions
## 2017

(1.) Consider $F = F_0 - hm + \tilde{a}_2 t m^2 + a_4 m^4$

Equil. magnetization:

$$\left.\frac{\partial F}{\partial m}\right|_{\bar{m}} = 0 \Rightarrow h = 2\tilde{a}_2 t \bar{m} + 4 a_4 \bar{m}^3$$

$$\chi^{-1} = \frac{\partial h}{\partial \bar{m}} = 2\tilde{a}_2 t + 12 a_4 \bar{m}^2, \text{ or}$$

$$\chi = \frac{1}{2\tilde{a}_2 t + 12 a_4 \bar{m}^2}.$$

Note that in zero field,

$$\begin{cases} \bar{m} \to 0 & t = 0^+, \\ \bar{m} \to \sqrt{-\dfrac{\tilde{a}_2 t}{2a_4}} & t = 0^-. \end{cases} \qquad t = \frac{T - T_c}{T_c} \text{ is the reduced } T$$

Then

$$\begin{cases} \chi(t \to 0^+) = \dfrac{1}{2\tilde{a}_2 t} \sim t^{-1}, \\ \chi(t \to 0^-) = \dfrac{1}{2\tilde{a}_2 t - 12 a_4 \dfrac{\tilde{a}_2 t}{2a_4}} = -\dfrac{1}{4\tilde{a}_2 t} \sim t^{-1} \end{cases}$$

Thus $\gamma_{mf} = 1$ & $\dfrac{\chi(t \to 0^+)}{\chi(t \to 0^-)} = \dfrac{1/2\tilde{a}_2|t|}{-1/4\tilde{a}_2(-|t|)} = 2$, as desired.

(2.)

$$\mathcal{H} = -J \sum_i S_i S_{i+1} \quad , \quad S_i = (-1, 0, 1)$$

The partition function is

$$Z = \sum_{\{S\}} e^{\beta J (S_0 S_1 + S_1 S_2 + \ldots + S_{N-1} \underset{\substack{\uparrow \\ S_N \text{ by} \\ \text{periodic BCs}}}{S_0})} \;\textcircled{=}$$

$$\textcircled{=} \sum_{\{S\}} e^{\beta J S_0 S_1} e^{\beta J S_1 S_2} \ldots e^{\beta J S_{N-1} S_0} .$$

It is clear that $T_{i,i+1} = e^{\beta J S_i S_{i+1}}$ should work:

$$T = \begin{array}{c} \begin{pmatrix} e^{\beta J} & 1 & e^{-\beta J} \\ 1 & 1 & 1 \\ e^{-\beta J} & 1 & e^{\beta J} \end{pmatrix} \begin{array}{c} -1 \\ 0 \\ 1 \end{array} \\ \begin{array}{ccc} -1 \quad & \quad 0 \quad & \quad 1 \end{array} \end{array}$$

The eigenvalues are:

$$\begin{cases} \lambda_0 = e^{\beta J} - e^{-\beta J} , \\ \lambda_{1,2} = \frac{1}{2} \left( 1 + e^{\beta J} + e^{-\beta J} \pm e^{-\beta J} \times \right. \end{cases}$$

$$\left. \times \sqrt{1 + 11 e^{2\beta J} + e^{4\beta J} - 2 e^{\beta J} - 2 e^{3\beta J}} \right) .$$

It turns out that $\lambda_1$ (with "+" in front of $e^{-\beta J}\sqrt{\ldots}$) is the largest eigenvalue for all finite $\beta J$. Therefore,

$$f = -k_B T \log \lambda_1 .$$

Note that

$$8 + (1 - 2\cosh(\beta J))^2 = e^{-2\beta J}(11 e^{2\beta J} + e^{4\beta J} + 1 - 2e^{3\beta J} - 2e^{\beta J}),$$ such that

$$\lambda_1 = \frac{1}{2}\left(1 + 2\cosh(\beta J) + \sqrt{8 + (1-2\cosh(\beta J))^2}\right),$$

and

$$\underline{\underline{f = -k_B T \log\left(1 + 2\cosh(\beta J) + \sqrt{8 + (1-2\cosh(\beta J))^2}\right) + k_B T \log 2 .}}$$

$\underline{T \to 0}$: $f = \lim_{\beta\to\infty}\left[-\frac{1}{\beta}\log(e^{\beta J} + e^{\beta J})\right] = -J,$

as expected in the low-T regime.

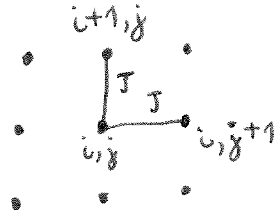
$\underline{T \to \infty}$: $f = \lim_{\beta \to 0}\left[-\frac{1}{\beta}\log\left(\frac{6}{2}\right)\right] \sim -k_B T \log 3,$

as expected in the hi-T, entropy-dominated regime.

(3.) The original partition function can be written as

$$Z = \sum_{\{s\}} e^{K \sum_{i,j} s_{i,j}(s_{i+1,j} + s_{i,j+1})},$$

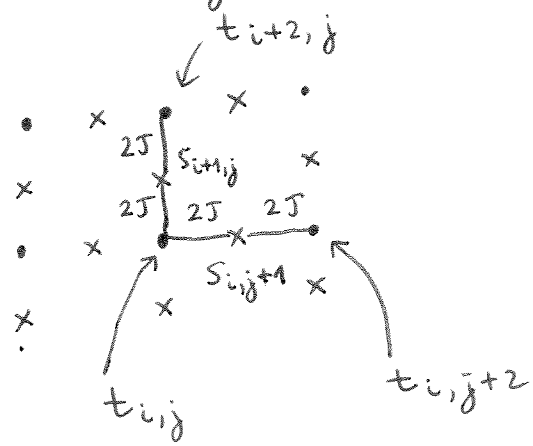
where $K = \beta J$ and the $i,j$ sum is over all the sites in the original lattice:




The MK transform replaces this lattice with another one (so it's not an exact transform but rather an educated guess):

$$Z = \sum_{\{s,t\}} \prod_{i,j=..2,4,6,..} e^{2K t_{i,j}(s_{i+1,j} + s_{i,j+1}) \oplus}$$

$$\oplus\ 2K s_{i+1,j} t_{i+2,j} + 2K s_{i,j+1} t_{i,j+2}.$$



Here, $S_{i+1,j}$ & $S_{i,j+1}$ will be summed over in the decimation step, as follows:

$$Z = \sum_{\{t\}} \prod_{i,j=..2,4,6...} \Big\{ e^{2K t_{i+2,j} + 2K t_{i,j+2} + 4K t_{i,j}} +$$

$$+ e^{-2K t_{i+2,j} - 2K t_{i,j+2} - 4K t_{i,j}} +$$

$$+ e^{2K t_{i+2,j} - 2K t_{i,j+2}} + e^{-2K t_{i+2,j} + 2K t_{i,j+2}} \Big\}.$$

Relabel the spins:

$$Z = \sum_{\{t\}} \prod_{i,j} \Big\{ e^{2K t_{i+1,j} + 2K t_{i,j+1} + 4K t_{i,j}} +$$

$$+ e^{-2K t_{i+1,j} - 2K t_{i,j+1} - 4K t_{i,j}} +$$

$$+ e^{2K(t_{i+1,j} - t_{i,j+1})} + e^{-2K(t_{i+1,j} - t_{i,j+1})} \Big\}.$$

On the other hand, we expect $Z = \sum_{\{t\}} \prod_{i,j} e^{K' t_{i,j}(t_{i+1,j} + t_{i,j+1})}$.

This leads to 9 equations (some of them the same) for

$t_{i,j} = \pm 1$, $\quad t_{i,j+1} = \pm 1$, $\quad t_{i+1,j} = \pm 1$

Focusing on the (1, 1, 1) & (1, −1, 1) equations,

we obtain:

$$\begin{cases} e^{8K} + e^{-8K} + 2 = e^{2K'}, & (1) \\ 2(e^{4K} + e^{-4K}) = 1. & (2) \end{cases}$$

Dividing (2) by (1) & setting

$$\begin{cases} x' = e^{-2K'}, \\ x = e^{-2K} \end{cases}$$

, we get:

$$x' = 2\,\frac{\frac{1}{x^2} + x^2}{\frac{1}{x^4} + x^4 + 2} = \underline{\underline{\frac{2x^2}{1+x^4}}}, \text{ as desired}$$

Clearly, $x^* = 0$ & $x^* = 1$ are fixed points. The other real fixed point is $x^* = 0.544$ (has to be found numerically).

This corresponds to $K^* = 0.304$, which is quite a bit off the exact solution: $K^*_{exact} = 0.441$.