

Lecture 18

[RG: general]

What are the general lessons of the RG approach?

Consider $\bar{H} = H/k_B T$ - reduced Hamiltonian

$$\text{RG transform: } \bar{H}' = \hat{R} \bar{H}$$

↑
RG operator, decreases

the # dof from $N \rightarrow N'$

(e.g. by grouping spins)

$$\text{Then } b^d = \frac{N}{N'} > 1$$

↑
scale factor of the RG transform

$$\text{Essential condition: } Z_{N'}(\bar{H}') = Z_N(\bar{H})$$

Then reduced free en. per spin is

$$\bar{f}(\bar{H}') = b^d \bar{f}(\bar{H})$$

$$\text{Likewise, } \vec{r}' = b^{-1} \vec{r}$$
$$\vec{q}' = b \vec{q}, \text{ etc.}$$

Note also that at fixed points,

$$\bar{H}' = \bar{H} \equiv \bar{H}^*$$

$$\text{also, } \ell_b' = b^{-1} \ell_b$$

But we expect self-similarity:

$$\ell_b' = \ell_b = \ell_b^* \Rightarrow \ell_b^* = \infty$$

at criticality

↑

Flows in prm space

Consider $\vec{H}(\vec{\mu})$

vector of parameters like J, H (or K, h)

Under RG transform,

$$\vec{\mu}' = \hat{R} \vec{\mu} \quad (*)$$

At a fixed point, $\vec{\mu}' = \vec{\mu}^* \equiv \vec{\mu}^*$

Close to a fixed point,

$$\begin{cases} \vec{\mu} = \vec{\mu}^* + \delta\vec{\mu} \\ \vec{\mu}' = \vec{\mu}^* + \delta\vec{\mu}' \end{cases} \Rightarrow \delta\vec{\mu}' = A(\vec{\mu}^*) \delta\vec{\mu}$$

matrix ~~matrix~~ vector

Indeed, (*) gives

$$\vec{\mu}^* + \delta\vec{\mu}' = \hat{R}(\vec{\mu}^* + \delta\vec{\mu})$$

Taylor expansion of \hat{R}

$$\hat{R} \vec{\mu}^* = \vec{\mu}^*$$

$\hat{R} \rightarrow$ linear transform

A : eigenvalues λ_i ,
eigenvectors \vec{v}_i

Consider 2^V transforms:

$$\begin{aligned} A_1 A_2 &= U \begin{pmatrix} \lambda_1(b) & 0 \\ 0 & \lambda_2(b) \end{pmatrix} U^{-1} U \begin{pmatrix} \lambda_1(b) & 0 \\ 0 & \lambda_2(b) \end{pmatrix} U^{-1} \\ &= U \begin{pmatrix} \lambda_1(b) \lambda_1(b) & 0 \\ 0 & \lambda_2(b) \lambda_2(b) \end{pmatrix} U^{-1} \end{aligned}$$

2×2 matrices
for example \rightarrow

On the other hand,

$$A_1 A_2 = A = U \begin{pmatrix} \lambda_1(b^2) & 0 \\ 0 & \lambda_2(b^2) \end{pmatrix} U^{-1}$$

U is a matrix with \vec{v}_i as columns: $U = [\vec{v}_1, \vec{v}_2]$

(2)

$$\text{So, } \lambda_i(b) \lambda_i(b) = \lambda_i(b^2)$$

$$\downarrow$$

$$\lambda_i(b) = \underline{\underline{b^{y_i}}}$$

y_i : indep. of b
 \uparrow
 critical exponents

Thus, near a fixed point

$$\vec{\mu} = \vec{\mu}^* + \sum_i g_i \vec{v}_i$$

expansion in terms of eigenvectors of A

\uparrow
linear scaling fields

RG transform:

$$\vec{\mu}' = \vec{\mu}^* + A(\vec{\mu}^*) \underbrace{\sum_i g_i \vec{v}_i}_{\vec{g}_i} = \vec{\mu}^* + \sum_i \underbrace{g_i b^{y_i}}_{g'_i} \vec{v}_i$$

$$A(\vec{\mu}^*) \vec{v}_i = \lambda_i \vec{v}_i$$

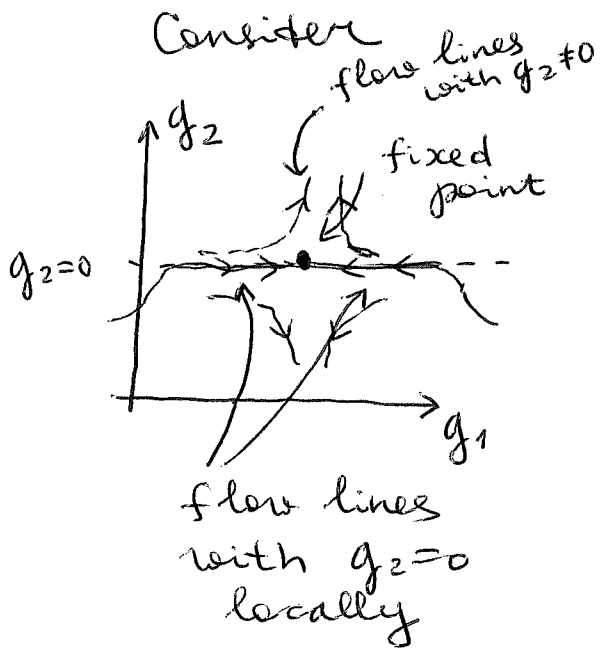
$$\text{So, } g'_i = \underline{\underline{b^{y_i} g_i}}$$

$y_i > 0 \Rightarrow$ the system is driven away from the fixed point \Rightarrow relevant variable (i)

$y_i < 0 \Rightarrow$ the system moves closer to the fixed point \Rightarrow irrelevant var.

$y_i = 0 \Rightarrow$ marginal var., 1st order expansion insufficient

Thus each fixed point is characterized by relevant & irrelevant vars. (scaling fields)



Let's say

$\left\{ \begin{array}{l} g_1 \text{ is irrelevant,} \\ g_2 \text{ is relevant} \end{array} \right.$

↓

$g_2=0$ is the critical surface (at least locally)

Critical surface:

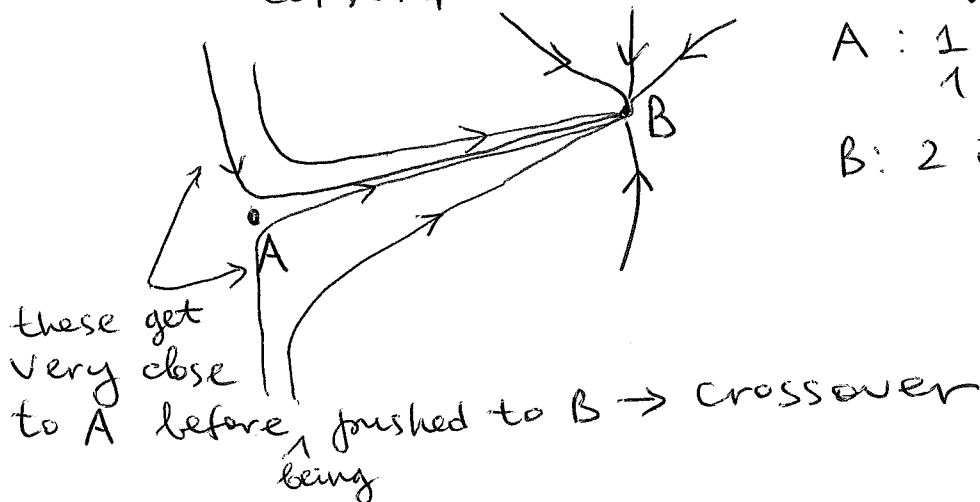
all points that flow into a critical point

all points on the crit. surf. have $\infty \xi$, since the fixed point has $\infty \xi$ & RG transforms can only decrease ξ .

Note that all systems that flow close to the critical point will be characterized by the same exponents $y_i \rightarrow$

→ universality

Consider another example:



A: 1 relevant, 1 irrelevant field

B: 2 irrelevant fields

So the procedure is:

- 1) Write down renormal'n flow eq's
e.g. $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$
- 2) Find fixed points $\begin{cases} x' = x = x^* \\ y' = y = y^* \end{cases}$
- 3) ~~Recall~~ Linearize around each fixed point: 2×2 matrix A

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \begin{pmatrix} \downarrow \\ \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

- Find λ_i & \vec{v}_i for A
Find $\gamma_i \rightarrow$ relev./irrelev. fields
determine
- 4) Find critical surfaces

Scaling & critical exponents

Recall that

$$\bar{f}(\vec{\mu}) = b^{-d} \bar{f}(\vec{\mu}')$$

Then $\bar{f}(g_1, g_2, g_3, \dots) = b^{-d} \bar{f}(b^{\gamma_1} g_1, b^{\gamma_2} g_2, b^{\gamma_3} g_3, \dots)$
close to the fixed point
Thus \bar{f} is a generalized ~~linear~~ homogeneous function:

$$f(\lambda^a x, \lambda^b y) = \lambda f(x, y)$$

e.g. $f(x, y) = x^3 + y^3$
 $\lambda f(x, y) = (\lambda^{1/3} x)^3 + (\lambda^{1/3} y)^3 \Rightarrow \begin{cases} a = 1/3 \\ b = 1/3 \end{cases}$

Now recall that $C_H \sim \left(\frac{\partial^2 \bar{f}}{\partial t^2} \right)_{h=0} \sim |t|^{-d}$
at zero field
(Table 2.3)

Assume that $g_1 = t$, $g_2 = h$ are relevant & all others irrelevant, then

$$\begin{cases} t = \frac{T - T_c}{T_c}, \\ h = \frac{H}{k_B T} \end{cases}$$

$$\bar{f}(t, h) \sim b^{-d} \bar{f}(b^{y_1} t, b^{y_2} h)$$

Can set $g_3 \rightarrow 0, g_4 \rightarrow 0, \dots$

Then $\left(\frac{\partial^2 \bar{f}}{\partial t^2} \right)_{h=0} \sim b^{-d+2y_1} \bar{f}_{tt}(b^{y_1} t, 0)$
 choose $h=0$ (zero field specific heat)
 $\bar{f}_{tt}(t, h=0)$

Now, choose $b^{y_1} |t| = 1$: $(b = |t|^{-\frac{1}{y_1}})$
 (can choose b freely)

$$\bar{f}_{tt}(t, 0) \sim |t|^{-\frac{1}{y_1} (2y_1 - d)} \bar{f}_{tt}(\pm 1, 0) =$$

$$= |t|^{\frac{d-2y_1}{y_1}} \bar{f}_{tt}(\pm 1, 0).$$

f'n of constant arguments

$$\text{So, } \underline{\underline{\alpha = 2 - \frac{d}{y_1}}}$$

Similarly, for zero-field magnetization

$$M \sim (-t)^\beta \Rightarrow \beta = \frac{d - y_2}{y_1}$$

Zero-field isothermal susceptibility

$$\chi_T \sim |t|^{-\gamma} \Rightarrow \gamma = \frac{2y_2 - d}{y_1}$$

But then $\alpha + 2\beta + \gamma = 2 - \frac{d}{y_1} + \frac{2d}{y_1} - \frac{2y_2}{y_1} +$

$$+ \frac{2y_2}{y_1} - \frac{d}{y_1} = \underline{\underline{2}}$$

Recall Rushbrooke inequality:

$$2 + 2\beta + \gamma \geq 2.$$

Turns out it's an equality (!)

[Note also that the exponents are the same below & above T_c

Finally,

Consider $M(t, h) \sim b^{-d+\gamma_2} M(b^{\gamma_1} t, b^{\gamma_2} h)$

$$M \sim \left(\frac{\partial F}{\partial h} \right)_T$$

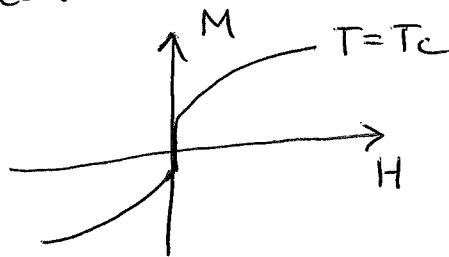
Choose $b^{\gamma_1} |t| = 1$:

$$M(t, h) \sim |t| \underbrace{\frac{(d-\gamma_2)}{\gamma_1}}_{\beta} M(\pm 1, h |t|^{-\underbrace{\frac{\gamma_2}{\gamma_1}}_{\beta\delta}})$$

where

$$\delta = \frac{\gamma_2}{d-\gamma_2}$$

Critical isotherm: $H \sim |M|^\delta \operatorname{sgn}(M)$



So, $M(t, h) \sim |t|^\beta M(\pm 1, \underbrace{h |t|^{-\beta\delta}}_{\bar{h}} \text{ scaled magnetic field})$

Define $\bar{m} = \frac{M(t, h)}{|t|^\beta} \rightarrow \bar{m} \sim M(\pm 1, \bar{h})$

Collapse of the data on 2 curves (one below T_c , one above T_c):

verified experimentally