Midterm solutions (2025)

1. (a) For a single classical particle,

$$Z_1 = \frac{1}{h^3}\int d^3q\, d^3p\, e^{-\beta H(p,q)} .$$

For ideal gas,

$$H(p,q) = \frac{\vec{p}^{\,2}}{2m}, \quad \text{leading to}$$

$$\vec{p} = (p_x, p_y, p_z)$$

$$Z_1 = \frac{V}{(2\pi\hbar)^3}\int d^3p\, e^{-\frac{\beta \vec{p}^{\,2}}{2m}} = V\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}$$

Since $\lambda = \sqrt{\frac{2\pi\hbar^2}{m k_B T}}$,

$\uparrow$ thermal wavelength

$$\underline{\underline{Z_1 = \frac{V}{\lambda^3}}} .$$

For N indistinguishable particles,

$$Z = \frac{Z_1^N}{N!} = \frac{V^N}{\lambda^{3N} N!} .$$

Finally, $F = -k_B T \log Z = -k_B T \Big[N \log V - 3N\log\lambda - \underbrace{\log N!}_{\simeq N\log N} \Big] =$

$$= \underline{\underline{-k_B T\Big[N\log V + \frac{3N}{2}\log T + N\,\underbrace{\text{const}(V,T)}_{\equiv C}\Big]}} .$$

(b) $$p = -\left(\frac{\partial F}{\partial V}\right)_T = \frac{Nk_BT}{V}, \text{ or}$$

$$\underline{\underline{pV = Nk_BT}} \leftarrow \text{ideal gas law}$$

(c) $$W = F_2 - F_1 \underset{\substack{\uparrow \\ T=\text{const}}}{=} Nk_BT \log\frac{V_1}{V_2} \underset{\substack{\uparrow \\ \text{EoS}}}{=} Nk_BT \log\frac{p_2}{p_1}.$$

Since $\mathcal{U} = \mathcal{U}(T)$ for ideal gas
and $T = \text{const}$,

$$\underline{\underline{Q = -W = Nk_BT \log\frac{V_2}{V_1}.}}$$

Alternatively, we can use

$Q = T(S_2 - S_1)$:

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = k_BN\left[\log V + \frac{3}{2}\log T + C_1\right] + $$

$$+ k_BT\,\frac{3N}{2T} = k_BN\left[\log V + \frac{3}{2}\log T\right] + C'.$$

Then $S_2 - S_1 \underset{\substack{\uparrow \\ T=\text{const}}}{=} k_BN \log\frac{V_2}{V_1}$ and

$Q = Nk_BT \log\frac{V_2}{V_1}$, consistent with above

(2.) For N HOs, the energy levels are labeled by $\{n_k\}$, where $k = 0, 1, 2, \ldots$ and $\sum_{k=0}^{\infty} n_k = N$. (*)

The energy of each state is given by $E\{n_k\} = \sum_{k=0}^{\infty} \hbar\omega_0 (k + \frac{1}{2}) n_k$.

Working in the grand-canonical ensemble (such that the constraint (*) is no longer enforced) and following the lecture notes, we immediately obtain:

$$\log \Sigma = - \sum_{k=0}^{\infty} \log\{1 - \underset{\text{fugacity}}{z}\, e^{-\beta\hbar\omega_0 (k+\frac{1}{2})}\},$$

$$\langle N \rangle = z \frac{\partial}{\partial z} \log \Sigma = \sum_{k=0}^{\infty} \frac{z\, e^{-\beta\hbar\omega_0 (k+\frac{1}{2})}}{1 - z\, e^{-\beta\hbar\omega_0 (k+\frac{1}{2})}}.$$