Lecture 18

[RG: general]

[What are the general lessons of the RG approach?]

Consider $\bar{\mathcal{H}} = \mathcal{H}/k_B T$ - reduced Hamiltonian

RG transform: $\bar{\mathcal{H}}' = \hat{R}\,\bar{\mathcal{H}}$

$\hat{R}$ ↑ RG operator, decreases the # dof from $N \to N'$ (e.g. by grouping spins)

Then $b^d = \frac{N}{N'} > 1$

b ↑ scale factor of the RG transform

Essential condition: $Z_{N'}(\bar{\mathcal{H}}') = Z_N(\bar{\mathcal{H}})$

Then reduced free en. per spin is

$\bar{f}(\bar{\mathcal{H}}') = b^d\,\bar{f}(\bar{\mathcal{H}})$

Likewise, $\vec{r}\,' = b^{-1}\vec{r}$

$\vec{q}\,' = b\vec{q}$, etc.

Note also that at fixed points,

$\bar{\mathcal{H}}' = \bar{\mathcal{H}} \equiv \bar{\mathcal{H}}^*$

also, $\xi' = b^{-1}\xi$

But we expect self-similarity:

$\xi' = \xi = \xi^* \Rightarrow \xi^* = \infty$ at criticality

Flows in prm space

Consider $\overline{\mathcal{H}}(\vec{\mu})$

vector of parameters like J, H (or K, h)

Under RG transform,

$$\vec{\mu}' = \hat{R}\vec{\mu} \qquad (*)$$

At a fixed point, $\vec{\mu}' = \vec{\mu} \equiv \vec{\mu}^*$

Close to a fixed point,

$$\begin{cases} \vec{\mu} = \vec{\mu}^* + \delta\vec{\mu}, \\ \vec{\mu}' = \vec{\mu}^* + \delta\vec{\mu}' \end{cases} \Rightarrow \delta\vec{\mu}' = \underset{\text{matrix}}{A(\vec{\mu}^*)}\,\underset{\text{vector}}{\delta\vec{\mu}}$$

Indeed, $(*)$ gives

$$\vec{\mu}^* + \delta\vec{\mu}' = \underbrace{\hat{R}(\vec{\mu}^* + \delta\vec{\mu})}_{}$$

Taylor expansion of $\hat{R}$

$$\hat{R}\vec{\mu}^* = \vec{\mu}^*$$

$\hat{R} \to$ linear transform

A: eigenvalues λ_i,
eigenvectors $\vec{v}_i$

Consider 2 RG transforms:

$$A_1 A_2 = \mathcal{U}\begin{pmatrix} \lambda_1(b) & 0 \\ 0 & \lambda_2(b) \end{pmatrix}\mathcal{U}^{-1}\mathcal{U}\begin{pmatrix} \lambda_1(b) & 0 \\ 0 & \lambda_2(b) \end{pmatrix}\mathcal{U}^{-1} =$$

2x2 matrices for example

$$= \mathcal{U}\begin{pmatrix} \lambda_1(b)\lambda_1(b) & 0 \\ 0 & \lambda_2(b)\lambda_2(b) \end{pmatrix}\mathcal{U}^{-1}$$

On the other hand,

$$A_1 A_2 = A = \mathcal{U}\begin{pmatrix} \lambda_1(b^2) & 0 \\ 0 & \lambda_2(b^2) \end{pmatrix}\mathcal{U}^{-1}$$

$\mathcal{U}$ is a matrix with $\vec{v}_i$ as columns: $\mathcal{U} = [\vec{v}_1\ \vec{v}_2]$

So, $\lambda_i(b)\lambda_i(b) = \lambda_i(b^2)$

$$\Downarrow$$

$$\lambda_i(b) = b^{y_i}$$

y_i indep. of b

$\uparrow$ critical exponents

Thus, near a fixed point

$$\vec{\mu} = \vec{\mu}^* + \sum_i g_i \vec{v}_i$$

expansion in terms of eigenvectors of A

$\uparrow$ linear scaling fields

RG transform:

$$\vec{\mu}' = \vec{\mu}^* + A(\vec{\mu}^*)\underbrace{\delta\vec{\mu}}_{\sum_i g_i \vec{v}_i} = \vec{\mu}^* + \sum_i \underbrace{g_i b^{y_i}}_{g_i'} \vec{v}_i$$

$$A(\vec{\mu}^*)\vec{v}_i = \lambda_i \vec{v}_i$$

So, $g_i' = b^{y_i} g_i$

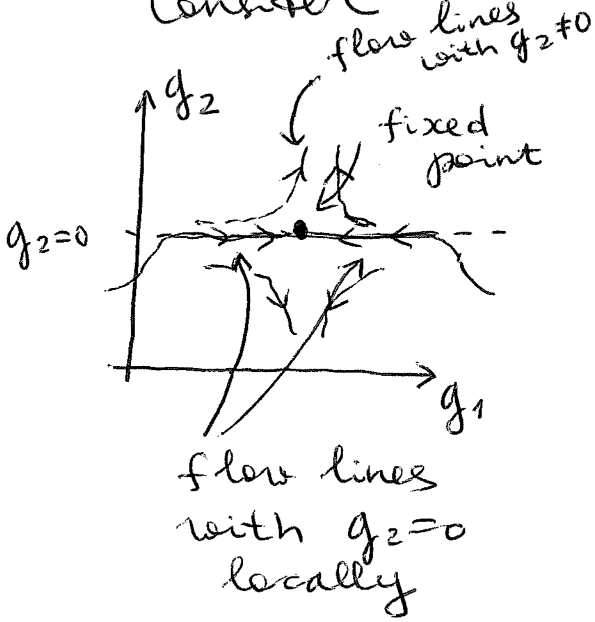
$y_i > 0 \Rightarrow$ the system is driven away from the fixed point $\Rightarrow$
$\Rightarrow$ relevant variable (i)

$y_i < 0 \Rightarrow$ the system moves closer to the fixed point $\Rightarrow$ irrelevant var.

$y_i = 0 \Rightarrow$ marginal var., 1st order expansion insufficient

Thus each fixed point is characterized by relevant & irrelevant vars. (scaling fields)

Consider



Let's say

$\begin{cases} g_1 \text{ is irrelevant,} \\ g_2 \text{ is relevant} \end{cases}$

$\downarrow$

$g_2 = 0$ is the critical surface (at least locally)

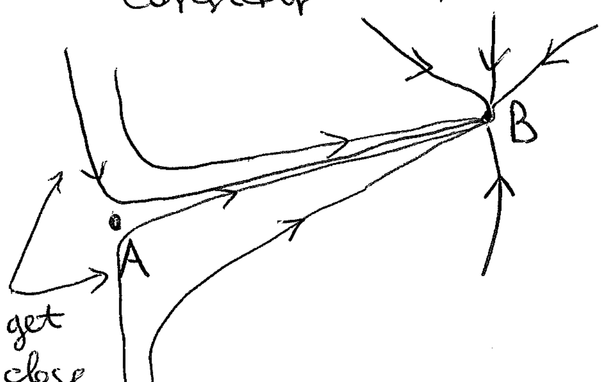
Critical surface: all points that flow into a critical point

All points on the crit. surf. have ∞ ξ, since the fixed point has ∞ ξ & RG transforms can only <u>decrease</u> ξ.

Note that all systems that flow close to the critical point will be characterized by the same exponents y_i →

→ <u>universality</u>

Consider another example:



A : 1 relevant, 1 irrelevant field

B: 2 irrelevant fields

these get very close to A before being pushed to B → crossover

So the procedure is:

1) Write down renormal'n flow eq's e.g. $\begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$

2) Find fixed points $\begin{cases} x' = x = x^* \\ y' = y = y^* \end{cases}$

3) ~~[illegible]~~ Linearize around each fixed point:

2x2 matrix A

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}$$

Find λ_i & $\vec{v}_i$ for A

Find y_i → relev./irrelev. fields (determine)

4) Find critical surfaces

Scaling & critical exponents

Recall that

$$\bar{f}(\vec{\mu}) = b^{-d} \bar{f}(\vec{\mu}')$$

Then close to the fixed point $\bar{f}(g_1, g_2, g_3, \dots) = b^{-d} \bar{f}(b^{y_1} g_1, b^{y_2} g_2, b^{y_3} g_3, \dots)$

Thus $\bar{f}$ is a generalized ~~[illegible]~~ homogeneous function:

$$f(\lambda^a x, \lambda^b y) = \lambda f(x,y)$$

E.g. $f(x,y) = x^3 + y^3$

$$\lambda f(x,y) = (\lambda^{1/3} x)^3 + (\lambda^{1/3} y)^3 \Rightarrow \begin{cases} a = 1/3 \\ b = 1/3 \end{cases}$$

Now recall that $C_H \sim \left(\frac{\partial^2 \bar{f}}{\partial t^2} \right)_{h=0} \underbrace{\sim |t|^{-\alpha}}_{\text{at zero field (Table 2.3)}}$

Assume that $g_1 = t$, $g_2 = h$ are relevant & all others irrelevant, then

$$\begin{cases} t = \frac{T - T_c}{T_c}, \\ h = \frac{H}{k_B T} \end{cases} \qquad \bar{f}(t, h) \sim b^{-d} \bar{f}(b^{y_1} t, b^{y_2} h)$$

$\Rightarrow$

Can set $g_3 \to 0$, $g_4 \to 0, \dots$

Then
$$\underbrace{\left(\frac{\partial^2 \bar{f}}{\partial t^2}\right)_{h=0}}_{= \bar{f}_{tt}(t, h=0)} \sim b^{-d+2y_1} \bar{f}_{tt}(b^{y_1} t, 0)$$

$\leftarrow$ choose $h=0$ (zero field specific heat)

Now, choose $b^{y_1}|t| = 1$: $\left(b = |t|^{-\frac{1}{y_1}}\right)$

(can choose b freely)

$$\bar{f}_{tt}(t, 0) \sim |t|^{-\frac{1}{y_1}(2y_1 - d)} \bar{f}_{tt}(\pm 1, 0) =$$

$$= |t|^{\frac{d - 2y_1}{y_1}} \underbrace{\bar{f}_{tt}(\pm 1, 0)}_{\text{f'n of constant arguments}}.$$

So, $\underline{\underline{\alpha = 2 - \frac{d}{y_1}}}$

Similarly, for zero-field magnetization
$$M \sim (-t)^{\beta} \Rightarrow \beta = \frac{d - y_2}{y_1}$$

zero-field isothermal susceptibility
$$\chi_T \sim |t|^{-\gamma} \Rightarrow \gamma = \frac{2y_2 - d}{y_1}$$

But then
$$\alpha + 2\beta + \gamma = 2 - \underline{\frac{d}{y_1}} + \underline{\frac{2d}{y_1}} - \cancel{\frac{2y_2}{y_1}} + \\ + \cancel{\frac{2y_2}{y_1}} - \underline{\frac{d}{y_1}} = \underline{\underline{2}}.$$

Recall Rushbrooke inequality:

$$\alpha + 2\beta + \gamma \geq 2.$$

Turns out it's an equality (!)

[Note also that the exponents are the same below & above T_c

Finally,

Consider $M(t,h) \sim b^{-d+y_2} M(b^{y_1} t, b^{y_2} h)$

$$M \sim \left(\frac{\partial f}{\partial h}\right)_T$$

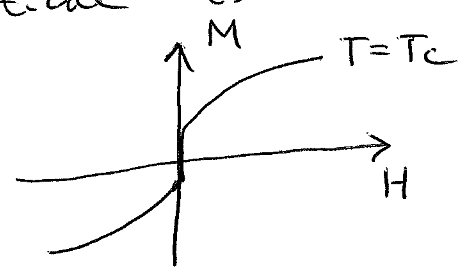
Choose $b^{y_1}|t| = 1$:

$$M(t,h) \sim |t|^{\underbrace{\frac{(d-y_2)}{y_1}}_{\beta}} M(\pm 1, h|t|^{-\underbrace{\frac{y_2}{y_1}}_{\beta\delta}}),$$

where

$$\delta = \frac{y_2}{d-y_2}$$

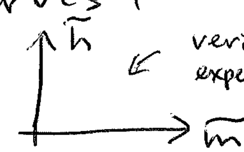
Critical isotherm: $H \sim |M|^{\delta} \operatorname{sgn}(M)$



So, $M(t,h) \sim |t|^{\beta} M(\pm 1, \underbrace{h|t|^{-\beta\delta}}_{\tilde{h}})$ scaled magnetic field

Define $\tilde{m} = \dfrac{M(t,h)}{|t|^{\beta}} \rightarrow \tilde{m} \sim M(\pm 1, \tilde{h})$

Collapse of the data on 2 curves (one below T_c, one above T_c):



verified experimentally