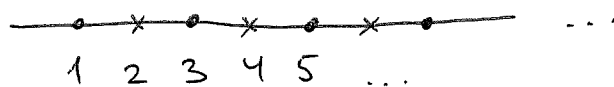
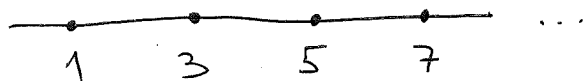


Lecture 17

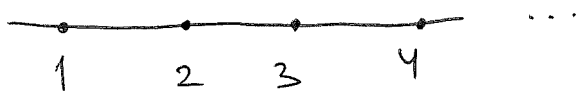
RG in 1D Ising model



original
lattice



renormalized
lattice



renorm'd &
renumbered
lattice

- 1) Write down recursive eq's
- 2) Extract fixed point structure + linearize around fixed points
- 3) Get a scaling form of free en.

Consider

$$\mathcal{H} = - \frac{K}{2} \sum_{i=0}^{N-1} (S_i S_{i+1} - \text{"}\beta \text{ included"} \times S_i S_{i-1}) - h \sum_{i=0}^{N-1} S_i - \sum_{i=0}^{N-1} C \quad \text{"some const"}$$

$b=2 \Rightarrow$ perform the trace in $\mathbb{Z}$ over even-numbered spins:

$$\mathcal{Z} = \sum_{\{S\}} \prod_{i=\dots 2, 4, 6, \dots} e^{K S_i (S_{i-1} + S_{i+1}) + h S_i + h \frac{S_{i-1} + S_{i+1}}{2} + 2C}$$

coupling
all terms with,

Indeed, consider S_3 :

"old way": $\frac{K}{2} S_2 S_3 + \frac{K}{2} (S_3 S_2 + S_3 S_4) + \frac{K}{2} S_4 S_3$)

"new way": $K S_2 S_3 + K S_4 S_3$

Likewise, for S_4 :

"old" $\frac{K}{2} S_3 S_4 + \frac{K}{2} (S_4 S_3 + S_4 S_5) + \frac{K}{2} S_4 S_5$

"new" $K(S_4 S_3 + S_4 S_5)$

Now do the partial trace: $\swarrow S_i = +1$

$$Z' = \sum_{\dots S_1, S_3, S_5, \dots} \prod_{i=\dots 2, 4, 6, \dots} \left\{ e^{K(S_{i-1} + S_{i+1}) + h + h \frac{S_{i-1} + S_{i+1}}{2} + 2C} + e^{-K(S_{i-1} + S_{i+1}) - h + h \frac{S_{i-1} + S_{i+1}}{2} + 2C} \right\} \quad \text{relabel spins consecutively}$$

$\nearrow S_i = -1$

$$\equiv \sum_{\{S\}} \prod_i \left\{ e^{(K + \frac{h}{2})(S_i + S_{i+1}) + h + 2C} + e^{-(K - \frac{h}{2})(S_i + S_{i+1}) - h + 2C} \right\}$$

$\uparrow$
"old" S_3, S_5
becomes "new" S_2, S_4 , etc.

Now, we require that

$$Z' = \sum_{\{S\}} \prod_i e^{K' S_i S_{i+1} + h' S_i + C'}$$

Then $h' \sum_i S_i \Rightarrow h' \sum_i \frac{S_i + S_{i+1}}{2}$

$$e^{K' S_i S_{i+1} + h' \frac{S_i + S_{i+1}}{2} + C'} =$$

$$= e^{(K + \frac{h}{2})(S_i + S_{i+1}) + h + 2C} + e^{-(K - \frac{h}{2})(S_i + S_{i+1}) - h + 2C}$$

We have 3 eq's :

$$\left\{ \begin{array}{l} S_i = 1, S_{i+1} = 1: e^{K'+h'+C'} = e^{2K+2h+2C} + e^{-2K+2C} \quad (1) \\ S_i = -1, S_{i+1} = -1: e^{K'-h'+C'} = e^{-2K+2C} + e^{2K-2h+2C} \quad (2) \\ S_i = 1, S_{i+1} = -1 \text{ or } S_i = -1, S_{i+1} = 1: e^{-K'+C'} = e^{h+2C} + e^{-h+2C} \quad (3) \end{array} \right.$$

Divide (1) by (2):

$$\begin{aligned} e^{2h'} &= \frac{e^{2K+2h+2C} + e^{-2K+2C}}{e^{2K-2h+2C} + e^{-2K+2C}} = \\ &= \frac{e^{2K+h} + e^{-2K-h}}{e^{2K-h} + e^{-2K+h}} \frac{e^h}{e^{-h}} = e^{2h} \frac{\cosh(2K+h)}{\cosh(2K-h)} \\ &= \end{aligned}$$

Likewise,

$$\left\{ \begin{array}{l} e^{4K'} = \frac{\cosh(2K+h) \cosh(2K-h)}{\cosh^2 h}, \\ e^{4C'} = e^{8C} \cosh(2K+h) \cosh(2K-h) \times \\ \quad \times \cosh^2(h) \end{array} \right.$$

Introduce $\begin{cases} w = e^{-4C}, \\ x = e^{-4K}, \\ y = e^{-2h} \end{cases}$

Then e.g.

$$\frac{1}{y'} = \frac{1}{y} \frac{\frac{1}{\sqrt{x}} \frac{1}{\sqrt{y}} + \sqrt{x} \sqrt{y}}{\frac{1}{\sqrt{x}} \sqrt{y} + \sqrt{x} \frac{1}{\sqrt{y}}} =$$

$$= \frac{1}{y} \frac{1+xy}{y+x}, \text{ or}$$

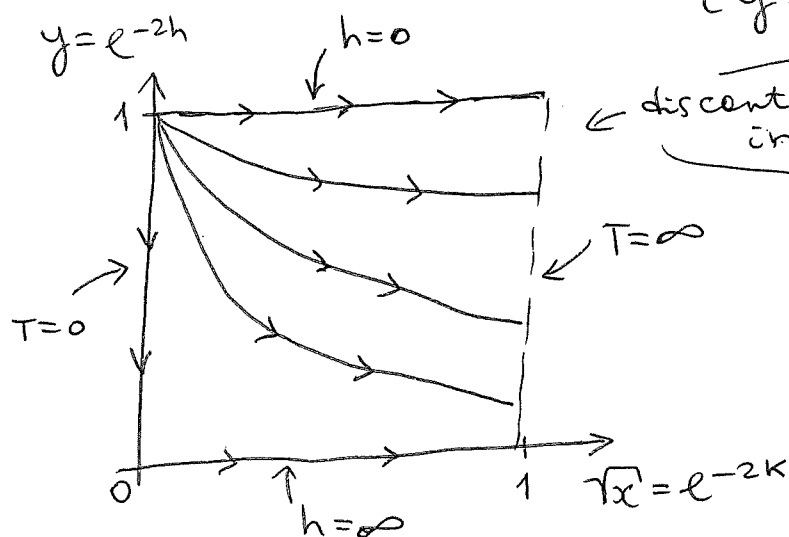
$$y' = y \frac{x+y}{1+xy} =$$

Likewise,
$$\begin{cases} \omega' = \frac{\omega^2 xy^2(1+y)^2}{(x+y)(1+xy)} \\ x' = \frac{x(1+y)^2}{(x+y)(1+xy)} \end{cases}$$

Note that x' & y' do not depend on ω .

So focus on the (x, y) plane first.

Consider $K > 0, h > 0 \Rightarrow \begin{cases} x \in [0, 1] \\ y \in [0, 1] \end{cases}$



Indeed,

$$\begin{cases} y^* = y^* \\ x^* = x^* = 1 \end{cases}$$

Line of fixed points at $\begin{cases} \sqrt{x}^* = 1 \\ 0 \leq y^* \leq 1 \end{cases} \leftarrow T = \infty \quad (\beta = 0)$

Paramagnetic fixed points:

all trajectories that start at $x > 0$ ($T > 0$) flow to them. This is to be expected since $T_c = 0$ in 1D Ising model & thus the system looks completely disordered on large enough scales.

Another fixed point: $(x^* = 0, y^* = 0)$ $\nwarrow T=0, h=\infty$
Indeed, you can never move off it...

Fully aligned configuration...

Finally, $(x^* = 0, y^* = 1) \leftarrow T=0, h=0$

$$\begin{cases} y^{*'} = y^{*2}, \leq y^{*'} = 1 \\ x^{*'} = x^* \frac{(1+y^*)^2}{y^*} = 4x^* \leq x^{*'} = 0 \end{cases}$$

Ferromagnetic fixed point...

Long-range correlations, critical behavior
Two variables, h & T , must be set to correct values before the system becomes critical. $\Leftarrow$ "least stable fixed point" since everything flows away from it.

Linearize the flow eq's around $(0, 1)$: $\begin{matrix} x^* & y^* \\ 0 & 1 \end{matrix}$

$$\begin{cases} x' \sim 4x, \\ (y-1)' \sim 2(y-1) \leq \epsilon' \sim 2\epsilon \end{cases} \quad \epsilon = y - y^* = y - 1$$

$\uparrow$
 $y' \sim 2y - 1$

Indeed, if $\begin{cases} x = \delta \\ y = 1 + \epsilon \end{cases} \Rightarrow y' \approx y(1 + \delta + \epsilon)(1 - \delta) =$
 $\begin{matrix} 1 + x & y \rightarrow 1 + \delta \end{matrix} \Rightarrow y' \approx y(1 + \delta + \epsilon - \delta) = y(1 + \epsilon) =$
 $\begin{matrix} 1 + \epsilon \end{matrix} \Rightarrow y' \approx 1 + 2\epsilon.$

$$y' = 1 + \epsilon' \Rightarrow \epsilon' \sim 2\epsilon, \text{ as above}$$

Total free en. must stay const:

$$\bar{f} = \beta f \quad (\text{reduced free en. per spin})$$

transforms as

$$\bar{f}' = b^d \bar{f} \quad \left(\begin{array}{l} b=2, \\ d=1, \text{ here} \end{array} \right)$$

More explicitly,

$$\bar{f} = b^{-1} \bar{f}(\underbrace{b^2 x}_{4x \sim x'}, \underbrace{b\epsilon}_{2\epsilon \sim \epsilon'})$$

We can iterate this to make b have any value (powers of 2 + analytic contin'n)

Hence choose $b^2 x = 1$:

$$\bar{f}(x, \epsilon) = \sqrt{x} \bar{f}\left(1, \frac{\epsilon}{\sqrt{x}}\right) = \sqrt{x} \tilde{f}\left(\frac{\epsilon}{\sqrt{x}}\right) \quad (*)$$

Recall that

$$f = -k_B T \log \left\{ e^{\beta J} \cosh(\beta H) + \sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}} \right\}$$

↑
from transfer matrix
↓
expand f in ϵ, x :

$$\bar{f}(x, \epsilon) = -K - \underbrace{\sqrt{x} \sqrt{1 + \frac{\epsilon^2}{4x}}}_{\text{singular part,}}$$

agrees with (*) if $\tilde{f}(z) = \sqrt{1 + z^2/4}$.

So, (*) predicts a scaling form near T_c

but does not give $\bar{f}(z)$ explicitly.

we can get the free en. as well;
consider the $H=0$ case for simplicity.

Consider

$$\bar{f}(x_0) - C_0^{\equiv 0} = 2^{-l} [\bar{f}(x_l) - C_l]$$

$\uparrow$ orig. prms $\uparrow$ const part split off explicitly $\uparrow$ renormalized prms

Note that

$$e^{4c'} = e^{8c''} \underbrace{f(k, h)}_{\cosh^2(2k)} \quad \begin{matrix} \equiv 0 \text{ here} \\ \downarrow \end{matrix}$$

$$C_l = 2C_{l-1} + \underbrace{\frac{1}{4} \log f(k_{l-1}, h_{l-1})}_{\text{"R}(K_{l-1})} \quad \begin{matrix} \equiv 0 \\ \downarrow \end{matrix}$$

$$\frac{1}{2} C_l = \underbrace{C_{l-1}}_{2C_{l-2} + R(K_{l-2})} + \frac{1}{2} R(K_{l-1}) \quad , \text{ or}$$

$$\frac{1}{4} C_l = C_{l-2} + \frac{1}{2} R(K_{l-2}) + \frac{1}{4} R(K_{l-1}), \text{ etc.}$$

In general, $2^{-l} C_l = C_0 + \sum_{k=0}^{l-1} 2^{-(k+1)} R(K_k)$

$\underbrace{C_0}_0$ $\underbrace{R(K_k)}_{\text{"}x_k\text{"}}$

$$\text{So, } \bar{f}(x_0) = 2^{-l} \bar{f}(x_l) - \sum_{k=0}^{l-1} 2^{-(k+1)} R(x_k)$$

for large l , end up
at a "trivial" fixed
point where $\bar{f}(x_l)$
can be calculated

$R(x_k)$ can be also computed along the flow lines ... $\Rightarrow$ get $\bar{f}(x_0)$ in the end.

1D Ising model: flow to $(y^*=1, x^*=1)$, $\uparrow$ no field! $\downarrow T=\infty$

a paramagnetic fixed point:

$$\bar{f}_*(x_e) = \log 2$$

as previously defined $\downarrow$

$$w_e = e^{-4C_e} \Rightarrow \bar{f}(x_e) - C_e = \log 2 + \log w_e^{1/4} =$$

$$= \log\left(\frac{w_e^{1/4}}{2}\right).$$

$$\frac{1}{w_e^{1/4}} = e^{C_e}$$

$$-\log w_e = C_e,$$

$$-C_e = \log w_e^{1/4}$$

Then $\bar{f}(x_0) = \lim_{l \rightarrow \infty} 2^{-l} \log\left(\frac{w_e^{1/4}}{2}\right).$

Now, use $(y=1)$

$$\begin{cases} w' = \frac{4w^2x}{(x+1)^2}, \\ x' = \frac{4x}{(x+1)^2}. \end{cases}$$

Define

$$\begin{cases} u = \frac{(wx)^{1/4}}{2}, \\ v = \frac{1-\sqrt{x}}{1+\sqrt{x}} = \tanh(k) \end{cases}$$

$$v^2 = \frac{(1-\sqrt{x})^2}{(1+\sqrt{x})^2} = \frac{1-2\sqrt{x}+x}{1+2\sqrt{x}+x}$$

$$v' = \frac{1-\sqrt{x'}}{1+\sqrt{x'}} = \frac{1 - \frac{2\sqrt{x}}{x+1}}{1 + \frac{2\sqrt{x}}{x+1}} = \frac{x+1-2\sqrt{x}}{x+1+2\sqrt{x}}$$

So, $v' = v^2$ (*)

Likewise,

$$u' = \frac{u^2(1+v)^2}{1+v^2} \quad (**)$$

$$l \rightarrow \infty \Rightarrow x_l \rightarrow 1 \Rightarrow \frac{u_l^{1/4}}{2} \rightarrow u_l.$$

$$\text{So, } \bar{f}(x_0) = \lim_{l \rightarrow \infty} 2^{-l} \log(u_l)$$

Iterate (*) & (**):

$$\log u_l = 2 \log u_{l-1} + 2 \log(1+v_{l-1}) - \log(1+v_{l-1}^2) =$$

$$= 2 \log \left[\frac{u_{l-2}^2 (1+v_{l-2})^2}{1+v_{l-2}^2} \right] +$$

$$+ 2 \log(1+v_{l-2}^2) - \log(1+v_{l-2}^4) =$$

$$= 2^2 \log u_{l-2} + 2^2 \log(1+v_{l-2}) -$$

$$- \log(1+v_{l-2}^{2^2}) = \dots =$$

$$= 2^l \log u_0 + 2^l \log(1+v_0) - \log(1+v_0^{2^l})$$

Recall that $v = \tanh K < 1$ for finite T :

$$\lim_{l \rightarrow \infty} (1+v_0^{2^l}) = 1$$

$$\text{Thus } \bar{f}(x_0) = \log u_0(1+v_0)$$

$$\text{Now, } u_0 = \frac{(\log x_0)^{1/4}}{2} = \frac{1}{2} (\underbrace{e^{-4x_0}}_1 e^{-4\kappa})^{1/4} =$$

$$= \frac{1}{2} e^{-\kappa}.$$

$$\text{So, } \bar{f}(x_0) = \log \left[\frac{1}{2} e^{-\kappa} \left(1 + \frac{e^{\kappa} - e^{-\kappa}}{e^{\kappa} + e^{-\kappa}} \right) \right] =$$

$$= \log \left[\frac{1}{2} e^{-\kappa} \underbrace{\frac{2e^{\kappa}}{e^{\kappa} + e^{-\kappa}}}_{\cosh(\kappa)} \right] = -\log [2 \cosh(\kappa)].$$

From transfer matrix:

$$\bar{f}_{\substack{\uparrow \\ H=0}} = -\log \{ e^{\kappa} + e^{-\kappa} \} = -\log \{ 2 \cosh(\kappa) \},$$

Same result!