

Lecture 16

Transfer matrix techniques

Consider 1D Ising model: N spins

$$H = -J \sum_{i=0}^{N-1} S_i S_{i+1} - H \sum_{i=0}^{N-1} S_i$$

Periodic BCs: $S_0 = S_N$, irrelevant in the $N \rightarrow \infty$ limit

Then

$$Z = \sum_{\{S\}} e^{\beta J (S_0 S_1 + S_1 S_2 + \dots + S_{N-1} S_0) + \beta H (S_0 + S_1 + \dots + S_{N-1})} \quad (\equiv)$$

↑ sum over $S_i = \pm 1, \forall i$

$$(\equiv) \sum_{\{S\}} e^{\beta J S_0 S_1 + \beta H \frac{S_0 + S_1}{2}} e^{\beta J S_1 S_2 + \beta H \frac{S_1 + S_2}{2}} \dots$$

$$\dots e^{\beta J S_{N-1} S_0 + \beta H \frac{S_{N-1} + S_0}{2}} \quad (\equiv)$$

Define $T_{i,i+1} = e^{\beta J S_i S_{i+1} + \beta H \frac{S_i + S_{i+1}}{2}}$

$$T = \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta(J-H)} \end{pmatrix} \begin{matrix} S_i = 1 \\ S_i = -1 \end{matrix}$$

Consider $N=2$: $S_2 = S_0$

$$Z = \sum_{\{S\}} e^{\beta J S_0 S_1 + \beta H \frac{S_0 + S_1}{2}} e^{\beta J S_1 S_0 + \beta H \frac{S_1 + S_0}{2}} =$$

$$\sum_{\substack{S_0 = \pm 1 \\ S_1 = \pm 1}} e^{\beta J + \beta H} e^{\beta J + \beta H} + e^{\beta J - \beta H} e^{\beta J - \beta H} +$$

$$\begin{matrix} \nearrow & \uparrow \\ S_0 = 1, S_1 = 1 & S_0 = -1, S_1 = -1 \end{matrix}$$

$$+ e^{-\beta J} e^{-\beta J} + e^{-\beta J} e^{-\beta J} \quad \diamond$$

$\uparrow$ $S_0=1, S_1=-1$ $\uparrow$ $S_0=-1, S_1=1$

On the other hand, consider

$$T^2 = \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{+\beta(J-\bar{H})} \end{pmatrix} \begin{pmatrix} e^{\beta(J+H)} & e^{-\beta J} \\ e^{-\beta J} & e^{+\beta(J-\bar{H})} \end{pmatrix} =$$

$$= \begin{pmatrix} e^{\beta(J+H)} e^{\beta(J+H)} + e^{-\beta J} e^{-\beta J} & e^{\beta(J+H)} e^{-\beta J} + e^{+\beta(J-\bar{H})} e^{-\beta J} \\ e^{\beta(J+H)} e^{-\beta J} + e^{+\beta(J-\bar{H})} e^{-\beta J} & e^{-\beta J} e^{-\beta J} + e^{+\beta(J-\bar{H})} e^{+\beta(J-\bar{H})} \end{pmatrix}$$

$$\diamond \text{Tr}(T^2)$$

general case

$$\diamond \sum_{\{s\}} T^{0,1} T^{1,2} \dots T^{N-1,0} = \text{Tr}(T^N) = \sum_{i=0}^{2 \times 2 T_d} \lambda_i^N$$

λ_i - eigenvalue of T

- 1D $\checkmark$ q -state Potts model: $q \times q$ transfer matrix
- 1D Ising model with 1st & 2nd nb inter's: 4×4

So, consider an $n \times n$ T : $\lambda_0 \dots \lambda_{n-1}$ s.t.

$$T|u_i\rangle = \lambda_i |u_i\rangle \quad |\lambda_0| > |\lambda_1| > \dots > |\lambda_{n-1}|$$

free en. per spin

Then $f = -k_B T \lim_{N \rightarrow \infty} \frac{1}{N} \log Z =$

Can prove that λ_0 is non-degenerate & positive for transfer matrices

$$= -k_B T \lim_{N \rightarrow \infty} \frac{1}{N} \log \left\{ \lambda_0^N \left(1 + \sum_{i=1}^{n-1} \left(\frac{\lambda_i}{\lambda_0} \right)^N \right) \right\} =$$

$\lambda_i (i \neq 0)$ may be complex but (*) still holds

$$= -k_B T \log \lambda_0 \quad (*)$$

λ_0 is non-degenerate

Next, correlation f'n: [consider 1D nnb Ising model for simplicity]

$$T_R = \langle S_0 S_R \rangle - \langle S_0 \rangle \langle S_R \rangle$$

↑
thermal
average

$$T_R \sim e^{-R/\xi} \text{ when } R \text{ is large:}$$

(purely exp'l)

$$\xi^{-1} = \lim_{R \rightarrow \infty} \left\{ -\frac{1}{R} \log |T_R| \right\}$$

Consider $\langle S_0 S_R \rangle = \frac{\sum_{\{S\}} S_0 S_R e^{-\beta H}}{\sum_{\{S\}} e^{-\beta H}}$

$Z = \sum_{i=0}^{n-1} \lambda_i^N$

~~As an easy example, consider~~

$$\left[\langle S_0 \rangle = \sum_{\substack{S_0 = \pm 1 \\ S_1 = \pm 1}} S_0 e^{\beta J S_0 S_1 + \beta H \frac{S_0 + S_1}{2}} \right]$$

Introduce $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
a spin matrix

Note that $\langle S_0 | = \begin{pmatrix} 1 & 0 \end{pmatrix}$ are its eigenvector
 $\langle S_1 | = \begin{pmatrix} 0 & 1 \end{pmatrix}$

with eigenvalues $S_0 = 1$ & $S_1 = -1$ respectively

Also, recall that $S = \sum_{i=0,1} |S_i\rangle S_i \langle S_i|$

Likewise, $T = \sum_{i=0,1} |u_i\rangle \lambda_i \langle u_i|$

$$S_0, Z = \text{Tr}(T^N) = \sum_{i,j=0,1} \langle s_j | u_i \rangle \lambda_i^N \langle u_i | s_j \rangle.$$

Indeed, note that for any 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} a \\ c \end{pmatrix}} + \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\begin{pmatrix} b \\ d \end{pmatrix}} = a + d = \text{Tr}(A)$$

$$S_0, \text{Tr}(A) = \sum_{i=0,1} \langle s_i | A | s_i \rangle$$

Note also that

$$Z = \sum_{j=0,1} \sum_{i_0 \dots i_{N-1}} \langle s_j | u_{i_0} \rangle \lambda_{i_0} \overbrace{\langle u_{i_0} | u_{i_1} \rangle}^{\delta_{i_0 i_1}} \lambda_{i_1} \overbrace{\langle u_{i_1} | u_{i_2} \rangle}^{\delta_{i_1 i_2}} \dots \lambda_{i_{N-1}} \langle u_{i_{N-1}} | s_j \rangle$$

It's clear that to make

$\sum_{\{s\}} S_0 S_R e^{-\beta H}$ rather than Z , we need to consider

$$\sum_{k=0,1} \sum_{j=0,1} \sum_{i_0 \dots i_{N-1}} S_0 \underbrace{\langle s_j | u_{i_0} \rangle \lambda_{i_0} \langle u_{i_0} | u_{i_1} \rangle \dots}_{\text{explicitly label spins}} \times \langle u_{i_{N-1}} | s_k \rangle S_R \langle s_k | u_{i_R} \rangle \dots =$$

$$= \sum_{j,k} \sum_{i_0, i_1} S_0 \langle s_j | u_{i_0} \rangle \lambda_{i_0}^R \langle u_{i_0} | s_k \rangle S_R \langle s_k | u_{i_1} \rangle \times \lambda_{i_1}^{N-R} \langle u_{i_1} | s_j \rangle$$

Note that

$$\sum_k |s_k\rangle \langle s_k| = 1$$

Next, we realize that

$$\sum_j \langle u_{i_1} | S_j \rangle S_0 \langle S_j | u_{i_0} \rangle = \\ = \langle u_{i_1} | S_0 | u_{i_0} \rangle,$$

$$\sum_k \langle u_{i_0} | S_k \rangle S_R \langle S_k | u_{i_1} \rangle = \langle u_{i_0} | S_R | u_{i_1} \rangle$$

$$\begin{cases} i_0 \rightarrow i, \\ i_1 \rightarrow j \end{cases} \Rightarrow \sum_{\{S\}} S_0 S_R e^{-\beta H} = \sum_{i,j} \langle u_j | S_0 | u_i \rangle \times$$

$$\times \lambda_i^R \langle u_i | S_R | u_j \rangle \lambda_j^{N-R}.$$

Finally,

$$\langle S_0 S_R \rangle = \frac{\sum_{i,j} \langle u_j | S_0 | u_i \rangle \left(\frac{\lambda_i}{\lambda_0}\right)^R \langle u_i | S_R | u_j \rangle \left(\frac{\lambda_j}{\lambda_0}\right)^N}{\sum_k \left(\frac{\lambda_k}{\lambda_0}\right)^N}$$

$N \rightarrow \infty$: only $j=0, k=0$ terms survive

$$\lim_{N \rightarrow \infty} \langle S_0 S_R \rangle = \sum_{i=0,1} \left(\frac{\lambda_i}{\lambda_0}\right)^R \underbrace{\langle u_0 | S_0 | u_i \rangle}_{\langle S_0 \rangle} \underbrace{\langle u_i | S_R | u_0 \rangle}_{\langle S_R \rangle} = \\ = \langle u_0 | S_0 | u_0 \rangle \langle u_0 | S_R | u_0 \rangle + \\ + \left(\frac{\lambda_1}{\lambda_0}\right)^R \langle u_0 | S_0 | u_1 \rangle \langle u_1 | S_R | u_0 \rangle.$$

So, $T_R = \left(\frac{\lambda_1}{\lambda_0}\right)^R \langle u_0 | S_0 | u_1 \rangle \langle u_1 | S_R | u_0 \rangle$
depends on all eigenvalues
& eigenvectors of T

$\frac{n > 2}{\text{dim of the transfer matrix}}$: $T_R = \sum_{i \neq 0} \left(\frac{\lambda_i}{\lambda_0} \right)^R \langle u_0 | S_0 | u_i \rangle \langle u_i | S_R | u_0 \rangle$

Finally,

$$\zeta^{-1} = \lim_{R \rightarrow \infty} \left[-\frac{1}{R} \log \left\{ \left(\frac{\lambda_1}{\lambda_0} \right)^R \langle u_0 | S_0 | u_1 \rangle * \langle u_1 | S_R | u_0 \rangle \right\} \right] \in$$

$\nearrow$
i=1 term dominates

$$\equiv -\log \left(\frac{\lambda_1}{\lambda_0} \right) \quad \text{includes only } \lambda_0, \lambda_1$$

—○—

explicit results for 1D Ising model:

$$\lambda_{0,1} = e^{\beta J} \cosh(\beta H) \pm \sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}}$$

$$\langle u_0 | = \frac{\alpha_+ + \alpha_-}{2}$$

$$\langle u_1 | = \frac{\alpha_- - \alpha_+}{2}$$

$$\alpha_{\pm}^2 = \frac{1}{2} \left(1 \pm \frac{e^{\beta J} \sinh(\beta H)}{\sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}}} \right)$$

So, free en. per spin:

$$f = -k_B T \log(\lambda_0) = -k_B T \log \left\{ e^{\beta J} \cosh(\beta H) + \sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}} \right\}.$$

$$\beta \rightarrow \infty:$$

$$f = -k_B T \log \left\{ e^{\beta J} \frac{e^{\beta H}}{2} + e^{\beta J} \frac{e^{\beta H}}{2} \right\} =$$

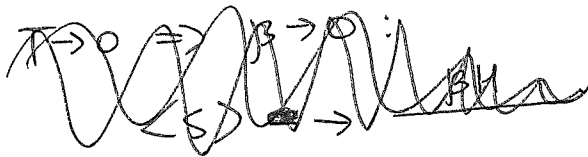
$$= -k_B T (\beta J + \beta H) = -J - H.$$

↑
just energy per spin

Magnetization:

$$\langle S \rangle = \langle u_0 | S | u_0 \rangle = \underbrace{\frac{1}{2} + \frac{1}{2}}_{\substack{\uparrow \\ \text{eigenvector} \\ \text{w/ largest eigenvalue}}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} + \\ \frac{1}{2} - \end{pmatrix}}_{\begin{pmatrix} \frac{1}{2} + \\ -\frac{1}{2} - \end{pmatrix}} \quad \textcircled{=}$$

$$\textcircled{=} \quad \cancel{\frac{1}{2} + \frac{1}{2}} \quad \frac{e^{\beta J} \sinh(\beta H)}{\sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}}}.$$



$$T=0 \Rightarrow \langle S \rangle = \frac{\sinh(\beta H)}{\sqrt{1 + \sinh^2(\beta H)}} = \tanh(\beta H)$$

↑
paramagnetic phase

$H \rightarrow 0$, T finite:

$$\langle S \rangle \rightarrow 0.$$

$$\underline{\text{But:}} \quad \lim_{H \rightarrow 0} \lim_{T \rightarrow 0} \langle S \rangle = \pm \frac{e^{\beta J} e^{\beta H/2}}{e^{\beta J} e^{\beta H/2}} = \pm 1$$

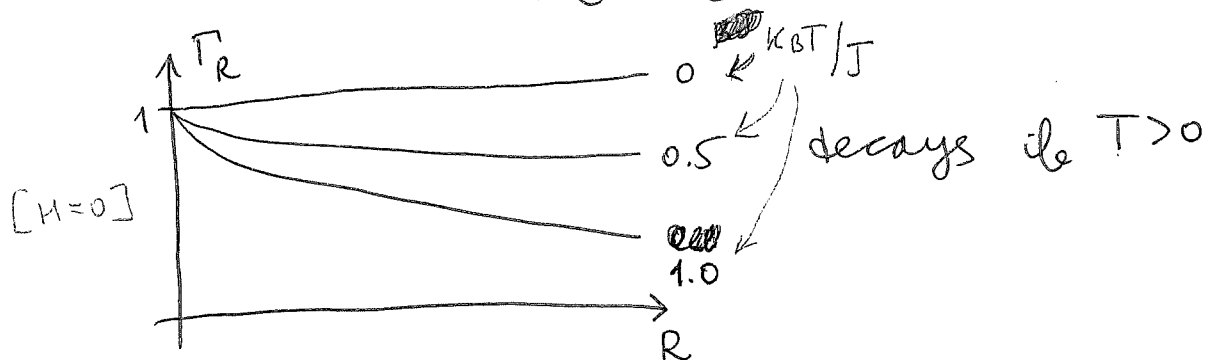
So, we have a phase transition
at $T_c = 0 \Rightarrow \langle S \rangle = 0$
goes to $\langle S \rangle = \pm 1$

Corr'n f'n $\begin{pmatrix} \lambda_- \\ \lambda_+ \end{pmatrix}$

$$\begin{aligned} T_R &= \left(\frac{\lambda_1}{\lambda_0} \right)^R \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\begin{pmatrix} \lambda_+ & \lambda_- \end{pmatrix}} \begin{pmatrix} \lambda_- \\ -\lambda_+ \end{pmatrix}^* \\ &= \left(\frac{\lambda_1}{\lambda_0} \right)^R \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\begin{pmatrix} \lambda_+ & \lambda_- \end{pmatrix}} \begin{pmatrix} \lambda_+ \\ \lambda_- \end{pmatrix} = \\ &= \left(\frac{\lambda_1}{\lambda_0} \right)^R [2\lambda_+ \lambda_-] [2\lambda_- \lambda_+] = \\ &= \left(\frac{\lambda_1}{\lambda_0} \right)^R 4 \lambda_+^2 \lambda_-^2 = \left(\frac{\lambda_1}{\lambda_0} \right)^R \left(1 - \frac{e^{\beta J} \sinh(\beta H)}{\sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}}} \right) \end{aligned}$$

$$\times \left(1 + \frac{e^{\beta J} \sinh(\beta H)}{\sqrt{\dots}} \right) = \left(\frac{\lambda_1}{\lambda_0} \right)^R \frac{e^{-2\beta J}}{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}}$$

$$H=0: T_R \rightarrow \left(\frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}} \right)^R = \tanh^R(\beta J)$$



Corr'n length

Finally,

$$\xi^{-1} = -\log\left(\frac{\lambda_1}{\lambda_0}\right) = -\log\left(\frac{e^{\beta J} \cosh(\beta H) - \sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}}}{e^{\beta J} \cosh(\beta H) + \sqrt{\dots}}\right)$$

Consider $H=0$
for simplicity $T \rightarrow 0$ $\beta \rightarrow \infty$
 H, J finite
" 0

$$\xi^{-1} \rightarrow \log \frac{e^{\beta J} e^{\beta H/2} - e^{\beta J} e^{\beta H/2}}{e^{\beta J} e^{\beta H/2} + e^{\beta J} e^{\beta H/2}} \rightarrow 0$$

$\beta \rightarrow \infty \Rightarrow$ phase transition

Indeed, $T_R(H=0) = \tanh^R(\beta J)$ means that

$$\chi^{-1} = \lim_{R \rightarrow \infty} \left[-\frac{1}{R} \log T_R \right] = -\log(\tanh(\beta J)) = \underline{\underline{R \log(\tanh(\beta J))}}$$

$\beta \rightarrow 0 (T \rightarrow \infty)$: $\tanh(\beta J) = \frac{e^{\beta J} - e^{-\beta J}}{e^{\beta J} + e^{-\beta J}} \rightarrow$
 $\rightarrow \beta J$, so that

$$\chi^{-1} \rightarrow -\log(\beta J) \rightarrow +\infty$$

$\chi \rightarrow 0$ at high $T \Rightarrow$ no correl's

$\beta \rightarrow \infty (T \rightarrow 0)$: $\tanh(\beta J) \rightarrow 1$, so that

$$\chi^{-1} \rightarrow 0, \quad \chi \rightarrow \infty \text{ as } T \rightarrow 0$$

$\swarrow$
phase transition (!)