Lecture 12

Ideal Fermi gas

Recall that $\frac{\lambda^3}{v} = f_{3/2}(z)$, where

$$v = \frac{V}{N}, \quad \lambda = \sqrt{\frac{2\pi\hbar^2}{mk_BT}}.$$

$$f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^2}{1 + z^{-1} e^{x^2}}.$$

$\underline{z \ll 1}$: $f_{3/2}(z) = z - \frac{z^2}{2^{3/2}} + \ldots$

$\underline{z \gg 1}$: define $\nu = \log z = \beta\mu$, then

$$f_{3/2}(z) = \frac{4}{\sqrt{\pi}} \int_0^\infty dx \, \frac{x^2}{e^{x^2-\nu}+1} = \frac{2}{\sqrt{\pi}} \int_0^\infty dy \, \frac{\sqrt{y}}{e^{y-\nu}+1} \,(=)$$

$$y = x^2 \Rightarrow dy = 2x\,dx, \quad dx = \frac{dy}{2\sqrt{y}}$$

$$(=) \frac{4}{3\sqrt{\pi}} \int_0^\infty dy \, \frac{y^{3/2} e^{y-\nu}}{(e^{y-\nu}+1)^2} \,(=)$$

$\int$ by parts

$$(=) \frac{4}{3\sqrt{\pi}} \int_0^\infty dy \, \frac{e^{y-\nu}}{(e^{y-\nu}+1)^2} \left[\nu^{3/2} + \frac{3}{2}\nu^{1/2}(y-\nu) + \ldots\right]$$

$$= \frac{4}{3\sqrt{\pi}} \int_{-\nu}^\infty dt \, \frac{e^t}{(e^t+1)^2} \left[\nu^{3/2} + \frac{3}{2}\nu^{1/2} t + \ldots\right].$$

$t = y - \nu$

can extend to $-\infty$, $O(e^{-\nu})$ contribution, ν large

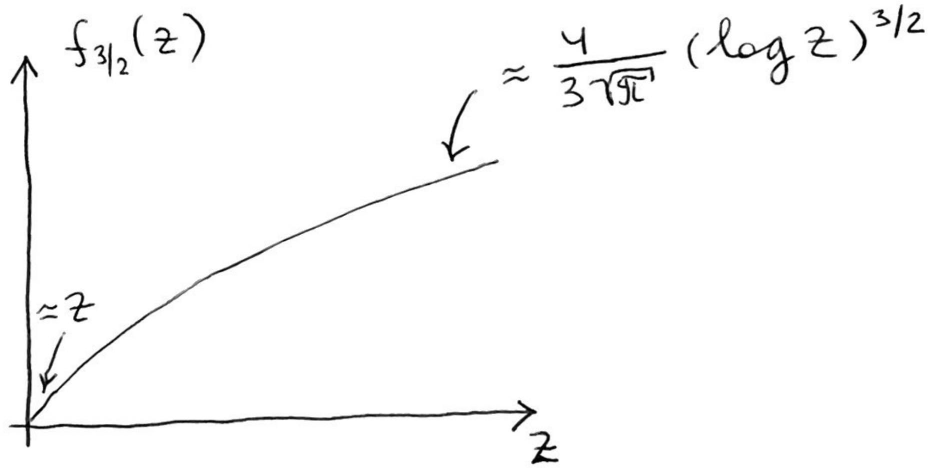
So, $f_{3/2}(z) = \dfrac{4}{3\sqrt{\pi}} \left[I_0 \nu^{3/2} + \dfrac{3}{2} I_1 \nu^{1/2} + \dots \right]$, where

$$I_n = \int_{-\infty}^{\infty} dt \, \frac{t^n e^t}{(e^t+1)^2}.$$

$$\begin{cases} I_n = 0, & \text{odd } n \\ \underset{n>0}{I_n} = (n-1)!\,(2n)\,(1-2^{1-n})\,\zeta(n), & \text{even } n>0 \\ I_0 = 1, & n=0 \end{cases}$$

In particular, $I_2 = 1! \times 4 \times (1-\frac{1}{2}) \underbrace{\zeta(2)}_{= \frac{\pi^2}{6}} = \dfrac{\pi^2}{3}$.

Then $f_{3/2}(z) = \dfrac{4}{3\sqrt{\pi}} \left[(\log z)^{3/2} + \dfrac{\pi^2}{8} (\log z)^{-1/2} + \dots \right]$



$z \nearrow$ as $\dfrac{\lambda^3}{v} \left(= f_{3/2}(z)\right) \nearrow$

Now, consider $\underbrace{\frac{\lambda^3}{v} \ll 1}$: $v^{1/3} \gg \lambda$, quantum effects small

In this case, $\frac{\lambda^3}{v} \simeq z - \frac{z^2}{2^{3/2}}$. $\Rightarrow z = \frac{\lambda^3}{v} + \mathcal{O}\left(\left(\frac{\lambda^3}{v}\right)^2\right)$

as $T \to \infty$ ($\beta \to 0$), $\lambda \to 0 \Rightarrow z = e^{\mu\beta} \sim p$, or

$$\underbrace{\mu\beta = \log p}_{\text{ideal gas}} + \text{const}$$

$$\langle n_{\vec{p}} \rangle = \frac{z e^{-\beta\epsilon_{\vec{p}}}}{1 - z e^{-\beta\epsilon_{\vec{p}}}} \underset{z \ll 1}{\simeq} \underbrace{\frac{\lambda^3}{v} e^{-\beta\epsilon_{\vec{p}}}}_{\text{Maxwell-Boltzmann}}.$$

Equation of state:

$$\frac{pv}{k_B T} = \frac{v}{\lambda^3} f_{5/2}(z) \underset{z \ll 1}{\simeq} \frac{v}{\lambda^3}\left[z - \frac{z^2}{2^{5/2}} + \dots\right] =$$

$$\overset{z = \frac{\lambda^3}{v}}{=} 1 - \frac{1}{2^{5/2}} \frac{\lambda^3}{v} + \dots$$

This is the EoS for ideal gas plus a small quantum correction.

Next, consider $\underbrace{\frac{\lambda^3}{v} >> 1}$: $v^{1/3} << \lambda$, quantum effects significant

In this case, $\frac{1}{v}\underbrace{\left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2}}_{\lambda^3} \simeq \frac{4}{3\sqrt{\pi}}(\log z)^{3/2}$, or

$$(+) \quad \underbrace{\log z}_{\underset{\beta\epsilon_F}{\parallel}} \simeq \left(\frac{2\pi\hbar^2}{mk_BT}\right)\left(\frac{3\sqrt{\pi}}{4v}\right)^{2/3},$$

$$\frac{2^{3/2}\,3}{4} = \frac{2^{3/2}\,3\times 2^{3/2}}{2^{3/2}\times 4} = \underline{\underline{\frac{6}{2^{3/2}}}}.$$

$$(*) \quad \epsilon_F = \underline{\frac{\hbar^2}{2m}\left(\frac{6\pi^2}{v}\right)^{2/3}}.$$

Fermi energy (chemical potential @ $T=0$)

$z = e^{\beta\epsilon_F}$ as $T\to 0$

Recall that $\langle n_{\vec{p}}\rangle = \frac{1}{1+z^{-1}e^{\beta\epsilon_{\vec{p}}}} =$

$$\overset{\text{low } T}{=} \frac{1}{1+e^{\beta(\epsilon_{\vec{p}}-\epsilon_F)}}.$$

Clearly, as $T\to 0$ $\langle n_{\vec{p}}\rangle = \begin{cases} 1, & \epsilon_{\vec{p}} < \epsilon_F \\ 0, & \epsilon_{\vec{p}} > \epsilon_F \end{cases}$

In the ground state, particles fill up all energy levels up to ϵ_F. In momentum space, the particles fill a sphere of some radius p_F $\Rightarrow$ the surface of the sphere is the <u>Fermi surface</u>.

$\epsilon_F = \frac{p_F^2}{2m}$

Now, assume particles of spin S, then

$$\underset{\substack{\uparrow \\ \text{each energy level contains } 2S+1 \text{ particles}}}{(2S+1)} \sum_{\vec{p}} \langle n_{\vec{p}} \rangle \underset{T\to 0}{\overset{\uparrow}{=}} N$$

Then $(2S+1) \dfrac{V}{(2\pi\hbar)^3} \displaystyle\int_0^{p_F} dp\, p^2\, 4\pi = N$, or

$$\frac{N}{V} = \frac{2S+1}{(2\pi\hbar)^3} \frac{4\pi}{3} p_F^3 \, , \quad \text{yielding}$$

$$\boxed{\epsilon_F = \frac{\hbar^2}{2m} \left(\frac{6\pi^2}{(2S+1)v} \right)^{2/3}}$$

consistent with Eq. (*) if $S=0$

Chemical potential:

$$\mu = k_B T \log z .$$

Recall that $\dfrac{\lambda^3}{v} = f_{3/2}(z) \simeq \dfrac{4}{3\sqrt{\pi}} \left[(\log z)^{3/2} + \right.$

$$\left. + \frac{\pi^2}{8} (\log z)^{-1/2} \right] = \frac{4}{3\sqrt{\pi}} (\log z)^{3/2} \left[1 + \underbrace{\frac{\pi^2}{8} (\log z)^{-2}}_{\text{small prm}} \right] .$$

Then $(\log z)^{3/2} \simeq \dfrac{\lambda^3}{v} \dfrac{3\sqrt{\pi}}{4} \left(1 - \dfrac{\pi^2}{8} (\underbrace{\log z}_{\substack{\shortparallel \\ \beta\epsilon_F \text{ to this order}}})^{-2} \right)$,

$$\log z \simeq \underbrace{\left(\frac{\lambda^3}{v} \frac{3\sqrt{\pi}}{4} \right)^{2/3}}_{\substack{\shortparallel \\ \beta\epsilon_F \text{ from Eq. (+) (by definition)}}} \left[1 - \underbrace{\frac{\pi^2}{8} \frac{2}{3}}_{\pi^2/12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right] .$$

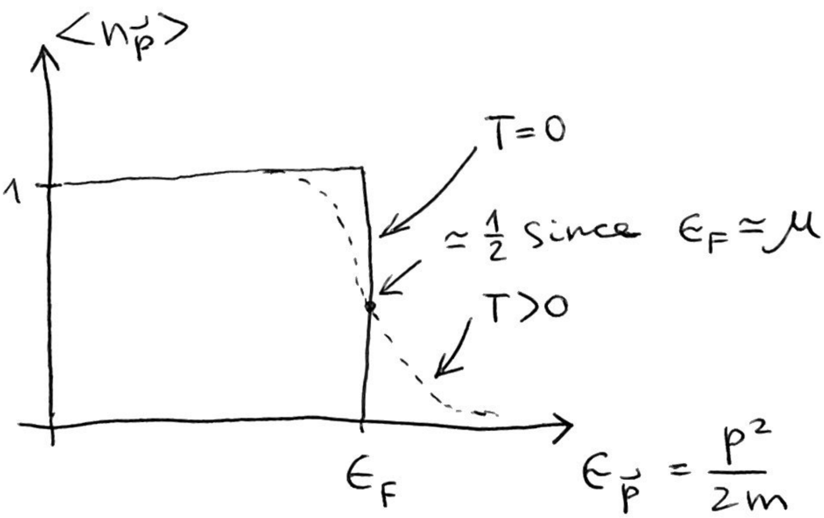
Finally,

$$\mu = k_B T \log z \approx \epsilon_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right] \qquad (**)$$

$\uparrow$ expansion in $\frac{k_B T}{\epsilon_F}$

Define the Fermi temperature T_F by $k_B T_F \equiv \epsilon_F$, then the expansion is valid for $\underline{T \ll T_F}$.

Next, $\langle n_{\vec{p}} \rangle = \dfrac{1}{e^{\beta(\epsilon_{\vec{p}} - \mu)} + 1}$, where μ is given by Eq. (**) in the $T \ll T_F$ regime.



Now, the internal energy

$$\mathcal{U} = \sum_{\vec{p}} \epsilon_{\vec{p}} \langle n_{\vec{p}} \rangle = \frac{V}{h^3} \frac{4\pi}{2m} \int_0^\infty dp\, p^4 \langle n_{\vec{p}} \rangle =$$

$$\underset{\substack{\uparrow \\ \text{by parts}}}{=} \frac{V}{4\pi^2 m \hbar^3} \int_0^\infty dp\, \frac{p^5}{5} \Big(- \frac{\partial}{\partial p} \underbrace{\langle n_{\vec{p}} \rangle}_{= \frac{1}{e^{\beta(\epsilon_{\vec{p}} - \mu)} + 1}} \Big) =$$

$$= \frac{\beta V}{20\pi^2 m^2 \hbar^3} \int_0^\infty dp\, \frac{p^6 e^{\beta(\epsilon_{\vec{p}} - \mu)}}{(e^{\beta(\epsilon_{\vec{p}} - \mu)} + 1)^2}.$$

$\frac{\partial}{\partial p} \langle n_{\vec{p}} \rangle$ is sharply peaked at $p = p_F$:
expand as in computing the $f_{3/2}(z)$
expansion for $z \gg 1$.

Then $$\mathcal{U} \simeq \frac{3}{5} N \epsilon_F \left[1 + \frac{5\pi^2}{12} \left(\frac{k_B T}{\epsilon_F} \right)^2 \right]. \quad (*)$$

Now, consider $$\sum_{|\vec{p}| < p_F} \frac{p^2}{2m} = \frac{V}{(2\pi\hbar)^3} \int_0^{p_F} dp\, p^4 \frac{4\pi}{2m} =$$

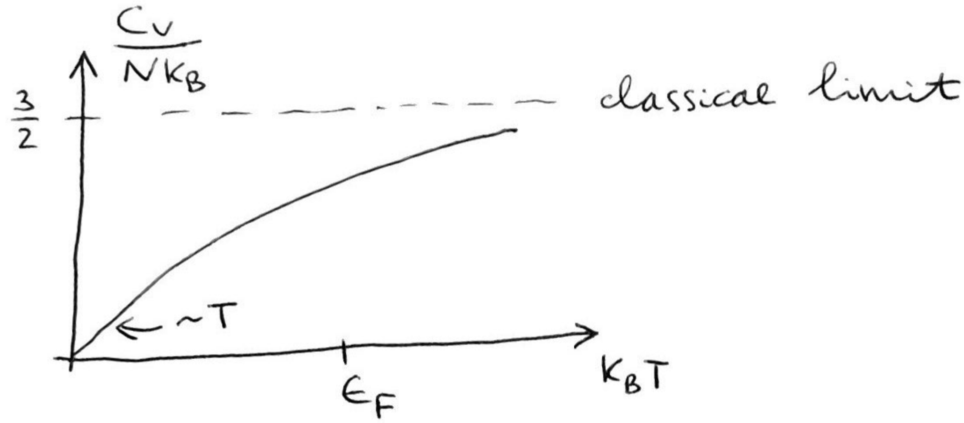
$$= \frac{V}{(2\pi)^2 \hbar^3 m} \frac{p_F^5}{5} = \frac{1}{5(2\pi)^2 \hbar^3 \cancel{m}} \Big(\underbrace{\frac{p_F^2}{2m}}_{\epsilon_F} 2\cancel{m} \Big) \underbrace{p_F^3 V}_{N \left(\frac{3}{4\pi}\right)(2\pi\hbar)^3} \ominus$$

$$\ominus\ N \epsilon_F \frac{3}{2} \frac{2}{5} = \underline{\underline{\frac{3}{5} N \epsilon_F}}.$$

So, the 1st term in Eq. (*) is the ground-state energy of the degenerate Fermi gas @ $T=0$.

Finally,

$$\frac{C_V}{Nk_B} \simeq \frac{3}{5}\epsilon_F \times \frac{5\pi^2}{12} 2 \frac{k_B T}{\epsilon_F} = \frac{\pi^2}{2}\frac{k_B T}{\epsilon_F}.$$



Only particles within $k_B T$ of ϵ_F can get excited at low T:

$$\frac{k_B T}{\epsilon_F} N \approx \# \text{ excited particles}$$

Then $$\Delta U \sim \frac{k_B T}{\epsilon_F} N k_B T = N\epsilon_F \left(\frac{k_B T}{\epsilon_F}\right)^2$$

Then $\frac{C_V}{Nk_B} \sim \frac{k_B T}{\epsilon_F}$, consistent with above

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The EoS is given by

$$p = \frac{2}{3}\frac{U}{V} \simeq \frac{2}{5}\frac{\epsilon_F}{v}\left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2\right].$$

$p \neq 0$ even if $T=0$ (!), due to Pauli exclusion principle.

Internal energy of Fermi gas

Lecture 12 extra notes

$$\mathcal{U} = \underbrace{\frac{\beta V}{20\pi^2 m^2 \hbar^3}}_{= G} \int_0^\infty dp \, \frac{p^6 e^{\beta(\epsilon_{\vec{p}} - \mu)}}{\left(e^{\beta(\epsilon_{\vec{p}} - \mu)} + 1\right)^2} \; \textcircled{=}$$

Recall that $\nu = \log z = \beta\mu$. (z: fugacity)

$$\underline{\epsilon_{\vec{p}} = \frac{p^2}{2m}}$$

$$y = \frac{\beta p^2}{2m} \Rightarrow dy = \frac{\beta}{m} p\, dp, \text{ or}$$

$$dp = \frac{m}{\beta} \frac{dy}{p}.$$

$$p = \left(\frac{2my}{\beta}\right)^{1/2}$$

$$\textcircled{=} \; G \int_0^\infty dy \, \frac{m}{\beta} \left(\frac{2my}{\beta}\right)^{5/2} \frac{e^{y-\nu}}{\left(e^{y-\nu} + 1\right)^2} \overset{t = y - \nu}{=}$$

$$= G\, 2^{5/2} \left(\frac{m}{\beta}\right)^{7/2} \int_{-\infty}^{\infty} dt \, (t+\nu)^{5/2} \frac{e^t}{\left(e^t + 1\right)^2} \; \diamond\!\!=$$

$-\infty$: extended from $-\nu$ to $-\infty$

$$\diamond\!\!= \; G\, 2^{5/2} \left(\frac{m}{\beta}\right)^{7/2} \int_{-\infty}^{\infty} dt \, \frac{e^t}{\left(e^t+1\right)^2} \left[\nu^{5/2} + \frac{5}{2}\nu^{3/2} t + \frac{15}{8}\nu^{1/2} t^2 + \ldots\right] = G\, 2^{5/2} \left(\frac{m}{\beta}\right)^{7/2} \times$$

$$\times \Big[\nu^{5/2} \underbrace{I_0}_{=1} + \frac{5}{2}\nu^{3/2} \underbrace{I_1}_{=0} + \frac{15}{8}\nu^{1/2} \underbrace{I_2}_{=\frac{\pi^2}{3}} + \ldots\Big].$$

Thus,

$$\mathcal{U} \simeq C\, 2^{5/2} \left(\frac{m}{\beta}\right)^{7/2} \left[(\log z)^{5/2} + \frac{15\pi^2}{24} (\log z)^{1/2} \right].$$

Now, $\log z \simeq \beta\epsilon_F \Big[1 - \underbrace{\frac{\pi^2}{12}\left(\frac{k_B T}{\epsilon_F}\right)^2}_{\text{to } \mathcal{O}\left(\left(\frac{k_B T}{\epsilon_F}\right)^2\right)}\Big]$, s.t.

lowest order only

$$\mathcal{U} \simeq C\, 2^{5/2} \left(\frac{m}{\beta}\right)^{7/2} \left[1 - \frac{\pi^2}{12} \cdot \frac{5}{2} \left(\frac{k_B T}{\epsilon_F}\right)^2 + \frac{5\pi^2}{8} \underbrace{\frac{(\beta\epsilon_F)^{1/2}}{(\beta\epsilon_F)^{5/2}}}_{\left(\frac{k_B T}{\epsilon_F}\right)^2} \right] \times$$

$$\times (\beta\epsilon_F)^{5/2}.$$

$$(\log z)^{5/2} \simeq (\beta\epsilon_F)^{5/2} \left[1 - \frac{\pi^2}{12} \frac{5}{2} \left(\frac{k_B T}{\epsilon_F}\right)^2 \right]$$

$$\uparrow$$

$$(1-x)^p = 1 - px$$

Next, consider

$$C\, 2^{5/2} \left(\frac{m}{\beta}\right)^{7/2} (\beta\epsilon_F)^{5/2} =$$

$$= \frac{\beta V 2^{1/2}}{5\pi^2 m^2 \hbar^3} m^{7/2} \frac{1}{\beta} \epsilon_F^{5/2} =$$

$$= \frac{N 2^{1/2}}{5\pi^2 \hbar^3 \underbrace{\frac{N}{V}}_{1/v}} m^{3/2} \epsilon_F^{5/2} = N \epsilon_F^{5/2} \frac{2^{1/2}}{5\pi^2} \frac{v\, m^{3/2}}{\hbar^3} =$$

Now, $\epsilon_F^{-3/2} = \left(\frac{2m}{\hbar^2}\right)^{3/2} \left(\frac{v}{6\pi^2}\right)$, or

$$\frac{m^{3/2}\vartheta}{\hbar^3} = \epsilon_F^{-3/2}\left(2^{-3/2}\,6\pi^2\right)$$

$$(=)\; N\epsilon_F^{5/2}\,\epsilon_F^{-3/2}\left(2^{-3/2}\,6\cancel{\pi^2}\right)\frac{2^{1/2}}{5\cancel{\pi^2}} =$$

$$= N\epsilon_F\,\frac{6}{2\times 5} = \underline{\underline{\frac{3}{5}N\epsilon_F}}.$$

Then $$U \simeq \frac{3}{5}N\epsilon_F\left[1-\frac{5\pi^2}{24}\left(\frac{k_BT}{\epsilon_F}\right)^2+\frac{5\pi^2}{8}\left(\frac{k_BT}{\epsilon_F}\right)^2\right] =$$

$$= \underline{\underline{\frac{3}{5}N\epsilon_F\left[1+\frac{5\pi^2}{12}\left(\frac{k_BT}{\epsilon_F}\right)^2\right]}}\,,$$ as desired

$$\frac{5}{8}^{(3} - \frac{5}{24} = \frac{10}{24} = \frac{5}{12}$$

$\boxed{\text{QED}}$