HW #5 solutions (2025)

(1.) (a) assume $\vec{H}_0 = H_0 \hat{z}$, then for a single spin

$$Z_1 = \int d\Omega\, e^{\beta \underbrace{\vec{H}_0 \cdot \vec{S}}_{H_0 \cos\theta}} = 2\pi \int_{-1}^{1} dx\, e^{\beta H_0 x} =$$

$$= \frac{4\pi}{\beta H_0} \sinh(\beta H_0).$$

Then $F_0 = -k_B T \log \overbrace{Z_1^N}^{= Z_0} = -k_B T N \log\left(\frac{4\pi}{\beta H_0} \sinh(\beta H_0)\right)$.

Next, $\langle \mathcal{H} - \mathcal{H}_0 \rangle_0 = -J \frac{Nz}{2} \langle \underbrace{S_z}_{\text{projection of } \vec{S} \text{ onto } z} \rangle_0^2 + N H_0 \langle S_z \rangle_0$.

The mean-field potential φ is given by

$$\varphi = F_0 + \langle \mathcal{H} - \mathcal{H}_0 \rangle_0, \quad \text{so that}$$

$$\frac{1}{N} \frac{\partial \varphi}{\partial H_0} = \overbrace{\frac{1}{N} \frac{\partial F_0}{\partial H_0}}^{-\langle S_z \rangle_0} - Jz \langle S_z \rangle_0 \frac{\partial \langle S_z \rangle_0}{\partial H_0} + \langle S_z \rangle_0 + H_0 \frac{\partial \langle S_z \rangle_0}{\partial H_0} = 0, \text{ or}$$

$$H_0 = Jz \langle S_z \rangle_0.$$

This generalizes to $\vec{H}_0 = Jz \langle \vec{S} \rangle_0$.

(b) Now,

$$F_{mf} = -k_B T \log Z_0 - \frac{JNz}{2} \langle S_z \rangle_0^2 + $$

$$+ N \langle S_z \rangle_0 \underbrace{Jz \langle S_z \rangle_0}_{H_0} = -k_B T \log Z_0 + \frac{JNz}{2} \langle S_z \rangle_0^2 .$$

In general,

$$F_{mf} = -k_B T \log Z_0 + \frac{JNz}{2} \langle \vec{S} \rangle_0 \cdot \langle \vec{S} \rangle_0 .$$

(2.) (a) Represent each philosopher by

$\sigma_i = \{-1, +1\}$; $i = 1, \dots, N$

σ_i ← spin; N ← # spins

Circular table → periodic BCs

Note that if $\sigma_i = +1$, $\sigma_{i-1} = \sigma_{i+1} = -1$ are not allowed.

(b) N = 2:

$$\left.\begin{array}{cc} - & - \\ - & + \\ + & - \end{array}\right\} M_N = 3$$

N = 3:

$$\left.\begin{array}{ccc} - & - & - \\ + & - & - \\ - & + & - \\ - & - & + \end{array}\right\} M_N = 4$$

N = 4:

$$\left.\begin{array}{l} - - - - \\ \text{4 states with one } + \\ - + - + \\ + - + - \end{array}\right\} M_N = 7$$

N = 5:

$$\left.\begin{array}{l} - - - - - \\ \text{5 states with one } + \\ \left.\begin{array}{l} + - + - - \\ + - - + - \\ - + - + - \\ - + - - + \\ - - + - + \end{array}\right\} \text{two } +\text{'s} \end{array}\right\} M_N = 11$$

(c) Note that

$$\underset{\overset{\shortparallel}{7}}{M_4} = M_3 + M_2 = 4 + 3,$$

$$\underset{\overset{\shortparallel}{11}}{M_5} = M_4 + M_3 = 7 + 4.$$

In general, $M_N = M_N(-1) + M_N(+1)$.

Now, $M_{N+1}(-1) = \underbrace{M_N}_{\text{just add } -1 \text{ @ pos. } N+1 \text{ to any } N\text{-spin configuration}} + \underbrace{M_{N-1}(+1)}$

$\underbrace{\cdots\uparrow}_{N-1}$ becomes $\underbrace{\cdots\uparrow\downarrow\uparrow}_{N+1}$

$M_{N+1}(+1) = \underbrace{M_{N-1}(-1)}$

$\underbrace{\cdots\downarrow}_{N-1}$ becomes $\underbrace{\cdots\downarrow\uparrow\downarrow}_{N+1}$

Therefore, $M_{N+1} = M_N + M_{N-1}$. $\quad (*)$

(d) The contribution of 2 neighboring spins to $\mathcal{H}$ is

$$\Delta E(\sigma, \sigma') = -J\sigma\sigma' - \frac{h}{2}(\sigma + \sigma').$$

with $J = -1$, $h = -2$ we obtain:

σ / σ'	$+1$	-1
$+1$	3	-1
-1	-1	-1

$\Leftarrow$ table of $\Delta E(\sigma, \sigma')$ values

All states that are allowed in the philosopher problem have the same ground-state energy $E_0(N) = -\underbrace{N}_{\substack{\#\text{ nnb} \\ \text{pairs in} \\ \text{1D chain}}}$.

The forbidden states have higher energies. Thus, it suffices to count the # ground states to find M_N.

(e) In the $\beta \to$ limit,

$$Z_N \to \sum_{\substack{\text{ground} \\ \text{states}}} e^{-\beta E_0(N)} = e^{-\beta E_0(N)} M_N.$$

excited states do not contribute

Thus, $M_N = \lim_{\beta \to \infty} e^{\beta E_0(N)} Z_N$.

Now, recall that

$$Z_N = \lambda_0^N + \lambda_1^N \,, \quad \text{where}$$

$$\lambda_{0,1} = e^{\beta J} \cosh(\beta H) \pm \sqrt{e^{2\beta J} \sinh^2(\beta H) + e^{-2\beta J}} \,.$$

with $J=-1$, $h=-2$ we have:

$$\lambda_{0,1} = e^{-\beta} \cosh(2\beta) \pm \sqrt{e^{-2\beta} \sinh^2(2\beta) + e^{2\beta}} \,.$$

In the $\beta \to \infty$ limit,

$$\lambda_{0,1} \to e^{-\beta} \frac{e^{2\beta}}{2} \pm \sqrt{e^{-2\beta} \frac{e^{4\beta}}{4} + e^{2\beta}} =$$

$$= e^{\beta} \frac{1}{2} \pm e^{\beta} \frac{\sqrt{5}}{2} = e^{\beta} \left(\frac{1 \pm \sqrt{5}}{2} \right) .$$

Finally,

$$M_N \underset{\beta\to\infty}{\longrightarrow} e^{-\beta N} \left[e^{\beta N} \left(\frac{1+\sqrt{5}}{2} \right)^N + e^{\beta N} \left(\frac{1-\sqrt{5}}{2} \right)^N \right] =$$

$$= \left(\frac{1+\sqrt{5}}{2} \right)^N + \left(\frac{1-\sqrt{5}}{2} \right)^N .$$

Note that $\overbrace{M_0 = 2, \quad M_1 = 1}^{\text{"non-physical states"}}$,

$$M_2 = \frac{1+2\sqrt{5}+5}{4} + \frac{1-2\sqrt{5}+5}{4} = 3 \,, \text{ etc.}$$

Finally, note that

$$M_N + M_{N-1} = \left(\frac{1+\sqrt{5}}{2}\right)^N + \left(\frac{1-\sqrt{5}}{2}\right)^N +$$

$$+ \left(\frac{1+\sqrt{5}}{2}\right)^{N-1} + \left(\frac{1-\sqrt{5}}{2}\right)^{N-1} =$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{N-1} \underbrace{\left(\frac{3+\sqrt{5}}{2}\right)}_{\left(\frac{1+\sqrt{5}}{2}\right)^2} + \left(\frac{1-\sqrt{5}}{2}\right)^{N-1} \underbrace{\left(\frac{3-\sqrt{5}}{2}\right)}_{\left(\frac{1-\sqrt{5}}{2}\right)^2} =$$

$= M_{N+1}$, consistent with (*).