HW #5 (2025)

(1.) **20 pt** Consider a Hamiltonian for a classical Heisenberg model:

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j \,, \quad \text{where } \vec{S}_i \text{ is a unit vector}$$

sum over nnb

Derive the mean-field equations using

$$\mathcal{H}_0 = -\vec{H}_0 \cdot \sum_{i=1}^{N} \vec{S}_i \quad \text{as the trial Hamiltonian}$$

N = # spins

(a) Derive a relation between $\vec{H}_0$ and $\langle \vec{S} \rangle_0$, where $\langle \ldots \rangle_0$ denotes a thermal average with respect to $\mathcal{H}_0$.

Hint: minimize the mean-field potential Φ; the answer should contain J and coord. # z.

(b) Write down an expression for the mean-field free energy F_{mf}.

(2.) **20 pt** [The Chinese philosophers problem]

Consider N philosophers sitting at a circular table. Each philosopher can either meditate (state -1) or eat (state $+1$). To eat, each philosopher needs 2 chopsticks. Chopsticks are shared with the 2 neighboring philosophers, one on the left and one on the right. Thus, if a philosopher is eating, his left & right neighbors cannot eat at the same time.

(a) Show that the problem is equivalent to N spins with periodic BCs:

$$\sigma_i \in \{-1, +1\}, \quad i = 1, \ldots, N$$

(b) Evaluate the total number of allowed configurations M_N for $N = 2, 3, 4, 5$

(c) Show that $M_N = M_{N-1} + M_{N-2}$

Hint: consider $M_N(\sigma) = \#$ acceptable configurations with $\sigma_N = \sigma$, where $\sigma \in \{-1, 1\}$

(d) Show that the problem reduces to counting the minimal energy states of a 1D Ising model

given by:

$$\mathcal{H} = \sum_{i=1}^{N} \left[-J\sigma_i\sigma_{i+1} - \frac{h}{2}(\sigma_i + \sigma_{i+1}) \right],$$

with $\sigma_{i+N} = \sigma_i$ (periodic BCs),

$J = -1$, $h = -2$.

Find the corresponding ground-state energy $E_0(N)$.

(e) Show that

$$M_N = \lim_{\beta\to\infty} e^{\beta E_0(N)} Z_N, \text{ where}$$

$$Z_N = \sum_{\{\sigma\}} e^{-\beta \mathcal{H}(\{\sigma\})}.$$

Find Z_N using the transfer matrix technique (OK to use results from the lecture notes) and evaluate M_N in this way. Does it match the expressions obtained above?